Lecture 13: Conditional probability

We ask the following question: suppose we know that the event E has occurred. How does this impact the probability of the event F. We write

$$P(F|E) \,=\, \text{the conditional probability of } F \text{ given } E$$

Example: Suppose a family has two children and suppose one of the children is a boy. What is the probability that both children are boys?

To answer this question we suppose that it is equally likely to have boys or girls. The the sample space for the children is $S = \{BB, BG, GB, GG\}$ where for example BG means that the first child is a boy and the second is a girl. We have

$$P(BB) = P(BG) = P(GB) = P(GG) \,=\, \frac{1}{4}$$

Let us consider the events

$$E = \{BG, GB, BB\}, \text{ one of the children is a boy}$$

and

$$F = \{BB\}, \text{ both children are boys}$$

Since all events are equally likely we if we know that F has occurred, we assign now the new probabilities 1/3 to all three events in F and thus we obtain

$$P(F|E) \,=\, \frac{1}{3}$$

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Definition of conditional probability: Given an event B, we assign new probabilities for each outcome in the sample space $p(i|E)$. Since we know that E has occurred then we must have

$$p(i|E) = 0\,, \qquad \text{for } i \in \overline{E}$$

and if $i \in E$ we require that the probabilities $p(i|E)$ have the same relative magnitude than $p(i)$. That is we require

$$p(i|E) \,=\, cp(i) \qquad \text{for } i \in E$$

But we must have

$$1 \,=\, \sum_{i\in E} p(i|E) \,=\, c \sum_{i \in E} p(i) \,=\, cP(E)$$

and thus

$$c = \frac{1}{P(E)}$$

So we define

Condititional probability of i given E : $\quad p(i\|E) = \frac{p(i)}{P(E)}, \quad i \in E$

If we consider an event F then we have

$$P(F|E) = \sum_{i \in F} P(i|E) = \sum_{i \in F \cap E} \frac{p(i)}{P(E)} = \frac{P(F \cap E)}{P(E)}$$

and so obtain

Condititional probability of F given E : $\quad P(F\|E) = \frac{P(F \cap E)}{P(E)}$

It is also useful to think of this formula in a different way: we can write

$$\boxed{P(F \cap E) = P(F|E)P(E)}$$

that is to compute the probability that both E and F occurs can be computed as the probability that E occurs time the conditional probability that F occurs given E.

Finally we give one more application of this formula: Suppose you want to compute the probability of an event F. Sometimes it is much easier to compute $P(F|E)$ or $P(F|\overline{E})$. We write

$$P(F) = P(F \cap E) + P(F \cap \overline{E}) = P(F|E)P(E) + P\left(F|\overline{E}\right) P\left(\overline{E}\right)$$

This formula is oftne very useful.

$$\textbf{Conditioning}: \; P(F) \;=\; P(F|E)P(E) + P(F|\overline{E})P(\overline{E})$$

More generally we can condition on a collection of n events provided they are pairwise disjoint and add up to all the sample space. If $S = E_1 \cup E_2 \cup \cdots E_n$ and the E_i are pariwse disjoint $E_i \cap E_j = \emptyset$, $i \neq j$ then we have

$$\textbf{Conditioning}: \; P(F) \;=\; P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + \cdots + P(F|E_n)P(E_n)$$

Example: the Monty's Hall problem At a game show the host hides a prize (say \$ 1 million) behind one of three doors and nothing behind the two remaining doors. The contestant picks one of three doors, say door 1, and then the game show host opens one of the reminaing door, say door 3 which has nothing behind it. The contestant is given the choice to either switch to door 2 or keep door 1. What should he do?

We will argue that he should switch to door 2 since there is a greater probability to find the prize behind door 2 than behind door 1.

To do this we assume that

- The contestant, upon choosing a door, has probability 1/3 to find the prize.
- The host show knows behind which doors the prize is and always opens an empty door. If he has two empty doors he can open then he chooses one of the two doors at random.

Let us start to analyze this problem when the the contestant has chosen door 1. We assume that

$$P(\text{prize door } i) \;=\; \frac{1}{3} \quad \text{for } i = 1, 2, 3.$$

If the prize is behind door 1 then the host show will open door 2 or door 3 each with probability 1/2. So we have

$$P(\text{prize door 1 } \textit{and} \text{ host door 2}) \;=\; \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P(\text{prize door 1 } \textit{and} \text{ host door 3}) \;=\; \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

On the other hand if the prize is behind door 2 or door 3, then the host has only door he can open, namely door 3 or door 2.

$$P(\text{prize door 2 } \mathit{and} \text{ host door 3}) = \frac{1}{3} \times 1 = \frac{1}{3}$$

$$P(\text{prize door 3 } \mathit{and} \text{ host door 2}) = \frac{1}{3} 1 = \frac{1}{3}$$

We have described all possibilities space starting from the fact that the contestant has already chose one door. Since the prize is behind any door with the same probability it does not matter which door is chosen.

Given that the host opens door 3 the probability to win the prize by keeping the door is the conditional probability

$$\begin{aligned} P(\text{Keep and win}) &= P(\text{prize door 1} \mid \text{host door 3}) \\ &= \frac{P(\text{prize door 1 } \mathit{and} \text{ host door 3})}{P(\text{ host door 3})} \\ &= \frac{\frac{1}{6}}{P(\text{ host door 3})} \end{aligned} \tag{1}$$

while

$$\begin{aligned} P(\text{Keep and loose}) &= P(\text{prize door 2} \mid \text{host door 3}) \\ &= \frac{P(\text{prize door 2 } \mathit{and} \text{ host door 3})}{P(\text{ host door 3})} \\ &= \frac{\frac{1}{3}}{P(\text{ host door 3})} \end{aligned} \tag{2}$$

It is therefore twice as likely to win by switching and so we have

$$\boxed{P(\text{Keep and win}) = \frac{1}{3} \quad P(\text{Keep and loose}) = \frac{2}{3}}$$

If one wishes to compute the probability that host opens door 3 then one can find it by conditioning on the location of the prize:

$$\begin{aligned} P(\text{ host door 3}) &= P(\text{host door 3} \mid \text{ prize door 1}) P(\text{prize door 1}) \\ &\quad + P(\text{host door 3} \mid \text{prize door 2}) P(\text{prize door 2}) \\ &\quad + P(\text{host door 3} \mid \text{prize door 3}) P(\text{prize door 3}) \\ &= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3} = \frac{1}{2} \end{aligned} \tag{3}$$

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Example: The game of crap: The game of crap is played as follows:

- Roll two dice and add the numbers obtained.
- If the total is 7 or 11 you win.
- If the total is 2, 3, 12 you loose.
- If the total is any other number (i.e., 4, 5, 6, 8, 9, 10) then this number is called "the point".
- Roll the pair of dice repeatedly until you obtain either "the point" or a 7.
- If you roll a 7 first you loose.
- If you roll "the point" first you win.

We compute the probability to win at this game. To do this we condition on the following events "first roll is 7 or 11", "the first roll is 2, 3, 12" "the point is 4", "the point is 5", etc.... We have then

$$\begin{aligned} P(\text{Win}) &= P(\text{Win}|\text{first roll 7 or 11})P(\text{first roll 7 or 11}) \\ &\quad + P(\text{Win}|\text{first roll 2 3 or 12})P(\text{first roll 2, 3, or 12}) \\ &\quad + \sum_{i \in \{4,5,6,8,9,10\}} P(\text{Win}|\text{point is } i)P(\text{point is } i) \end{aligned}$$

Most of these probabilities are easy to compute: The only one which requires some thought is $P(\text{Win}|\text{point is } i)$. Take for example the point to be 4. To compute this probability we argue that we roll the dice until we get a 4 or a 7 at which point the game stop. It does not matter how many times we roll the dice, the only thing which matters is that to win we need a 4 rather than a 7. So

$$P(\text{Win}|\text{point is 4}) = P(\text{roll a 4}|\text{roll a 4 or a 7}) = \frac{\frac{3}{36}}{\frac{3}{36} + \frac{6}{36}} = \frac{3}{9}$$

We leave it to the reader to verify that

$$P(\text{Win}) = 1 \times \frac{8}{36} + 0 \times \frac{4}{36} + \frac{3}{9} \times \frac{3}{36} + \frac{4}{10} \times \frac{4}{36} + \frac{5}{11} \times \frac{5}{36} + \frac{5}{11} \times \frac{5}{36} + \frac{4}{10} \times \frac{4}{36} + \frac{3}{9} \times \frac{3}{36} = .49293...$$

which shows that Crap was surely designed by a someone with a knowledge of probability...

Example: The game of roulette: We consider several variants of roulette.

- Las Vegas roulette has 38 numbers, 0 and 00 which are green and 1 to 36, half them being red and half of them being black. If you bet on red the probability to loose is $20/38 = 0.526315789$.

- Monte-Carlo roulette (1st version) has 37 numbers, 0 and 1 to 36, half them being red and half of them being black. If you roll a 0 then you are sent to prison (P1). At the next spin if you get a red you get your bet back (and nothing more), if you get black or 0, you loose. The probability to win is 18/37, and the probability to loose is obtained by conditioning on the first spin

$$\begin{aligned} P(\text{ lose }) &= P(\text{ lose } \mid \text{ black })P(\text{ black }) + P(\text{ lose } \mid P1\ )P(\ P1\ ) \\ &= 1 \times \frac{18}{37} + \frac{19}{37} \times \frac{1}{37} = 0.50036523 \end{aligned} \tag{4}$$

- Monte-Carlo roulette (2st version) is played as in the 1st version but with a second prison (P2). If you are in the first prison P1 you loose if the next spin is black and if the next spin is a 0 you are sent to the second prison P2. In P2 you loose if you get a black or 0 and are sent back to P1 if you get a red. The probability to lose is obtained by conditioning several times. First we have

$$P(\text{ lose }) = 1 \times \frac{18}{37} + P(\text{ lose } \mid P1\ )\frac{1}{37}$$

But if we are in P1 and condition on the next spin we have

$$P(\text{ lose } \mid P1\ ) = \frac{18}{37} + P(\text{ lose } \mid P2\ )\frac{1}{37}$$

and similarly if we are in P2

$$P(\text{ lose } \mid P2\ ) = \frac{19}{37} + P(\text{ lose } \mid P1\ )\frac{18}{37}$$

The last two equations can be combined to find the value of $P(\text{ lose } \mid P1\ )$

$$P(\text{ lose } \mid P1\ ) = \frac{18}{37} + \frac{1}{37}\left(\frac{19}{37} + P(\text{ lose } \mid P1\ )\frac{18}{37}\right)$$

which gives

$$P(\text{ lose } \mid P1\ ) = \frac{685}{1351}$$

and so

$$P(\text{ lose }) = \frac{18}{37} + \frac{1}{37}\frac{685}{1351} = 0.50019004\,.$$

Exercise 1: A coin is tossed three times. What is the probability that two heads occur given that

- The first outcome was a head.
- The first outcome was a tail.
- The first two outcomes were heads.
- The first tow outcomes were tails.
- The first outcome was a head and the third was a head.

Exercise 2: Three cards are drawn from a 52 card deck. What is the probability that the second card is an ace? (Condition on whether the first card is an ace or not.)

Exercise 3: Monty's Hall Suppose you have been watching Monty's Hall game for a very long time and have observed that the prize is behind door 1 45% of the time, behind door 2 40% of the time and behind door 3 15% of the time. The rest is as before.

1. When playing on the show you pick door 1 again and the host opens one empty door. Should you switch?
2. If you know you are going to be offered a switch would it be better to pick another door?