Week 3, Part 2: Linear difference equations

In this lecture we discuss how to solve *linear difference equations*.

First order homogeneous equation: You should think of the time being discrete and taking integer values $n = 0, 1, 2, \cdots$ and q_n describing the state of some system at time n. We consider an equation of the form

$$\boxed{\text{First order homogeneous} \quad aq_n + bq_{n-1} = 0}$$

where q_n, $n = 0, 1, 2, 3, \cdots$ are unknown and a and b are fixed constants. This equation is called a *first order homogeneous* equation. To solve it we rewrite it as

$$q_n = \left(\frac{-b}{a}\right) q_{n-1} = \alpha q_{n-1}$$

with $\alpha = -b/a$. This is easy to solve recursively

$$q_n = \alpha q_{n-1} = \alpha(\alpha q_{n-2}) \;=\; \alpha^2 q_{n-1} \;=\; \cdots \;=\; \alpha^n q_0$$

So if we are given q_0, e.g. the state of the system at time 0, then the state of the system at time n is given by $q_n = \alpha^n q_0$, i.e. this is a model for exponential growth or decay.

To summarize

$$\boxed{\text{The general solution of } aq_n + bq_{n-1} = 0 \text{ is } q_n = C\,(-b/a)^n}$$

Interest rate: A bank account gives an interest rate of 5% compounded monthly. If you invest \$1000, how much money do you have after 10 years? Since the interest is paid monthly we set

$$q_n = \text{ amount of money after } n \text{ months}$$

and since we get one twelfth of 5% every month we have

$$q_n \;=\; \left(1 + \frac{.05}{12}\right) q_{n-1} \;=\; \left(1 + \frac{1}{240}\right) q_{n-1} \;=\; \left(\frac{241}{240}\right) q_{n-1}$$

and so after 5 year we have with $q_0 = 1000$

$$q_{60} \;=\; \left(\frac{241}{240}\right)^{60} 1000 \;=\; 1283.35$$

First order inhomogeneous equation: Let us consider an equation of the form

$$\boxed{\text{First order inhomogeneous} \quad aq_n + bq_{n-1} = c_n,}$$

where c_n is a *given* sequence and q_n is unknown. For example we may take

$$c_n = c\,, \quad c_n = cn\,, \quad c_n = c\alpha^n\,.$$

This equation is called *inhomogeneous* because of the term c_n. The following simple fact is useful to solve such equations

Linearity principle: *Suppose $\hat{q}_n$ be a solution of the inhomogeneous $aq_n + bq_{n-1} = c_n$ and $\tilde{q}_n$ be a solution of the homogeneous equation $aq_n + bq_{n-1} = 0$ then $q_n + \tilde{q}_n$ is a solution of the inhomegenous equation $aq_n + bq_{n-1} = c_n$. Indeed we have*

$$\begin{aligned} a\hat{q}_n + b\hat{q}_{n-1} &= c_n \\ a\tilde{q}_n + b\tilde{q}_{n-1} &= 0 \end{aligned}$$

and thus adding the two equations give

$$a(\hat{q}_n + \tilde{q}_n) + b(\hat{q}_{n-1} + \tilde{q}_{n-1}) = c_n \tag{1}$$

To find the general solution of a first order homogeneous equation we need

- Find *one particular* solution of the inhomogeneous equation.
- Find the general solution of the homogeneous equation. This solution has a free constant in it which we then determine using for example the value of q_0.
- The general solution of the inhomogeneous equation is the sum of the particular solution of the inhomogeneous equation and general solution of the homogeneous equation.

Example: Solve

$$aq_n + bq_{n-1} = c$$

i.e., the inhomegenous term is $c_n = c$ i.e. constant. We look for a particular solution, and after some head scratching we try $q_n = d$ to be constant and find

$$ad + bd = c\,, \qquad \text{or } d = \frac{c}{a+b}$$

The general solution is then

$$q_n \;=\; C(-b/a)^n + \frac{c}{a+b}\,.$$

Example: Solve

$$2q_n - q_{n-1} = 2^n\,, \quad q_0 = 3$$

The solution of the homogenous is $q^n = C(1/2)^n$. To find a particular solution of the inhomogeneous problem we try an exponential function $q_n = D2^n$ with a constant D to be determined. Plugging into the equation we find

$$2D2^n - D2^{n-1} = 2^n$$

or after dividing by 2^{n-1}

$$4D - D = 2 \text{ or } D = \frac{2}{3}\,.$$

So the general solution is

$$q_n \;=\; C\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

and the initial condition gives $q_0 = 3 = C + \frac{2}{3}$ and so

$$q_n \;=\; \frac{7}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

More interest rate: A bank account gives an interest rate of 5% compounded monthly. If you invest invest initially \$1000, and add \$10 every month. How much money do you have after 10 years? Since the interest is paid monthly we set

$$q_n = \text{ amount of money after } n \text{ months}$$

and we have the equation for q_n

$$q_n = \left(1 + \frac{.05}{12}\right) q_{n-1} + 10 \;=\; \left(\frac{241}{240}\right) q_{n-1} + 10$$

For the particular solution we try $q_n = d$ and find

$$d = \frac{241}{240} d + 10$$

i.e., $d = -2400$. The general solution is then

$$q_n = C\left(\frac{241}{240}\right)^n - 2400$$

and $q_0 = 1000$ gives

$$q_n \;=\; 3400\left(\frac{241}{240}\right)^n - 2400$$

and so $q_{60} = 1963.41$

Second order homogeneous equation: We consider an equation where q_n depends on both q_{n-1} and q_{n-2}:

$$\boxed{\text{Second order homogeneous} \quad aq_n + bq_{n-1} + cq_{n-2} = 0\,.}$$

It is easy to see that we are given both q_0 and q_1 we can determine q_2, q_3, and so on.

Linearity Principle: *It is easy to verify that if $\tilde{q}_n$ and $\hat{q}_n$ are two solutions of the second order homogeneous equation. Then $C_1\tilde{q}_n + C_2\hat{q}_n$ is also a solution for any constant C_1, C_2.*

To find the general solution we get inspired by the homogeneous first order equation and look for solutions of the form

$$q_n = x^n$$

If we plug this into the equation we find

$$a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2}$$

and dividing by α^{n-2} give

$$ax^2 + bx + c \;=\; 0$$

We find (in general) two roots x_1 and x_2 and the general solution has the form

$$q_n = C_1 x_1^n + C_2 x_2^n$$

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Example: The **Fibonnaci sequence** is given by

$$q_n = q_{n-1} + q_{n-2}\,, \quad q_0 = 0, q_1 = 1$$

that is every term of the sequence is the sum of the two preceding terms. It is given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233\cdots$$

The golden ratio

$$\varphi \;=\; \frac{1+\sqrt{5}}{2} \;=\; 1.61803398875$$

occurs in the Fibonnacci sequence in the sense that for large n

$$\frac{q_{n+1}}{q_n} \approx \varphi \,.$$

For example $89/55 = 1.61818181818$, $144/89 = 1.61797752809$, $233/144 = 1.61805555556$, and so on... To see why it occurs we solve the second order difference equation: with $q_n = \alpha^n$ we find

$$\alpha^2 - \alpha - 1 \,=\, 0$$

or

$$\alpha \,=\, \frac{1 \pm \sqrt{5}}{2}$$

So the the general solution is

$$q_n \,=\, c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \,.$$

and with $q_0 = 0$ and $q_1 = 1$ we find

$$q_n \,=\, \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right] \,.$$

Since $|\frac{1-\sqrt{5}}{2}| < 1$ the second term is vanishingly small for large n so $q_n \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$.
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Second order inhomogeneous equation: We consider an equation of the form

$$\boxed{\text{Second order homogeneous} \quad aq_n + bq_{n-1} + cq_{n-2} = d_n \,.}$$

where q_n is unknown and d_n is a fixed sequence. As for first order equations we can solve such equations by

1. Solve the homogeneous equation $aq_n + bq_{n-1} + cq_{n-2} = 0$.
2. Find a particular solution of the inhomogeneous equation.
3. Write the general solution as the sum of the particular inhomogeneous equation plus the general solution of the homogeneous equation.

Example: Find the general solution of the second order equation $3q_n + 5q_{n-1} - 2q_{n-2} = 5$. For the homogeneous equation $3q_n + 5q_{n-1} - 2q_{n-2} = 0$ let us try $q_n = x^n$ we obtain the quadratic equation

$$3x^2 + 5x - 2 = 0 \text{ or } x = 1/3, -2$$

and so the general solution of the homogeneous equation is

$$q_n = C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

For a particular equation $3q_n + 5q_{n-1} - 2q_{n-2} = 5$ we try $q_n = D$ and find

$$3D + 5D - 2D = 5$$

i.e. $D = 5/6$ and so the general solution is

$$q_n = \frac{5}{6} + C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

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Exercise 1: Solve the following difference equations

1. $2q_n - 5q_{n-1} = 0$, $q_0 = 2$
2. $2q_n - 5q_{n-1} = 3$, $q_0 = 3$
3. $2q_{n+1} - 7q_n + 3q_{n-1} \;=\; 0, \quad q_0 = 1, q_1 = 2$
4. $2q_{n+1} - 7q_n + 3q_{n-1} \;=\; 2 + 2^n, \quad q_0 = 1, q_1 = 2$

Exercise 2: Your mortgage is a 30 year fixed rate mortgage at an (fixed) annual rate of 4% compounded monthly.

1. If you borrow \$150'000 today, what is the total amount of money will you pay back to the bank during the next 30 years?
2. You decide that you can make a down-payment of \$15000 and that \$1250 is the maximal monthly payment you are willing to commit to. What is the value of the most expensive house can you buy?

Exercise 3: Your retirement account has a fixed rate of 8% per year *paid yearly*. You start saving for retirement at age 30 with a target retirement age of 65 and \$0 in your saving account. Set-up and solve a suitable first order difference equations to answer the following questions (compute all interests and payment on a yearly basis).

1. Suppose you set aside \$500 every month. How much money will you have for your retirement?
2. You want to retire with \$500'000. How much should you save every month?
3. Assuming that your salary is going to increase 5 % per year during your lifetime you also decide you contribution should follow suit and your monthly contribution increase by 5% every year. If your starting contribution is \$500 every month how much money will you have saved at retirement age?
4. Assuming again that your contribution is increasing by 5% every year, what should your starting contribution be if if you want to reach \$500'000 by retirement age?

Exercise 4: After winning the \$ 200 million jackpot at the Powerball lottery you are given the choice to either receive the lump sum of \$ 100 million or to receive \$ 10 million per year for the next 20 years. You go first to a financial advisor who tells you that you can invest our money with him and receive an interest rate of α percent yearly (compounded annually). To compare your two options and because you are an extremely thrifty individual you decide to invest all your money for the next 20 years. Write down difference equations for the two options (denote by q_n the value of investment after n years in million dollars). For which interest rate α is the option of lump sum better?