

## Homework 7

1. Suppose you start with a fortune of 1 and throw a fair coin  $n$  times. A very rich and not very bright friend of yours pays you  $\frac{11}{5}k$  for a bet of  $k$  if the coin lands on heads and nothing if the coin lands on tail.
  - (a) Suppose you bet your entire fortune on each throw. What is your expected gain after 20 throws? What is the probability that you win nothing after 20 throws? What is the variance of your gain after 20 throws?
  - (b) What does Kelly's formula suggest you should do in this situation? What is your expected gain and its variance after 20 throws in that case?

2. **(Kelly betting with interest).**

- (a) You are given the possibility to make a risky bet which returns with probability  $p$ ,  $\$ \gamma$  for every dollar you bet. On the other hand if you do not bet you can invest your money in a bank and earn during the period of the bet an interest rate  $\alpha$ . You decide to follow the proportional play strategy and invest a portion  $f$  of your money in the bet and to invest the remaining portion  $(1 - f)$  in the bank.

Find the optimal  $f$  which will maximize the growth rate of your fortune.

- (b) Suppose you have the choice between a 100% safe investment returning 5% or a 90% safe investment returning 25% (this means that with probability 1/10 you will lose everything). Calculate how to invest your money using the Kelly strategy. Calculate the effective rate of return on your investment over the long term.

3. Consider the proportional play strategy, but now every single bet of 1 unit leads to three possible outcomes

$$P(\text{Win } 4) = 1/2 \quad P(\text{Lose } 1) = 1/4 \quad P(\text{Lose } 4) = 1/4.$$

Is it a superfair bet? What is the optimal proportion  $f^*$  of your fortune you should invest?

4. In a certain game if you bet 1 you are paid back  $\mu_1 > 1$  probability  $p$  and  $\mu_2$  with  $0 \leq \mu_2 < 1$  with probability  $1 - p$ .

- (a) For which values of  $\mu_1$  and  $\mu_2$  is the bet superfair?
- (b) Suppose your initial fortune is  $X_0$ , and you follow a proportional play strategy where you invest a proportion  $f$  of your fortune in the bet and keep a proportion  $(1 - f)$ . After  $n$  bets your fortune will take the form  $X_n = Q_n \cdots Q_1 X_0$  where  $Q_i$  are IID random variables. Describe the random variable  $Q_i$ ?

- (c) If you optimize your growth rate, show that in this case, the optimal Kelly fraction is

$$f^* = \min \left\{ \frac{p(\mu_1 - 1) + (1 - p)(\mu_2 - 1)}{(\mu_1 - 1)(1 - \mu_2)}, 1 \right\}$$

5. Suppose you are an investor in technology companies and you want to make quick money. You see thousands of new companies being brought to the stock market every year. It is very hard to predict in which direction the stock prices will move but for half of the companies the stock price will rise 80% during the first day after the stock is introduced while half of the other companies the stock price will fall 60% during this period. Your investment strategy is to buy such stock and sell them at the end of the day. Your initial fortune is \$10,000.

- (a) Suppose that for 1000 days you pick every day one stock at random and invest all your fortune in it. Estimate, using for example Chebyshev inequality, that you get more than 55% of the time stocks whose prices rise. If you assume that you will not pick more than 55% of the time stocks whose price rises, what is the best scenario for your fortune after 1000 days?
- (b) Use now instead a Kelly strategy where you invest a fixed proportion of your fortune every time. Using the results of the previous problem, what is the optimal Kelly fraction? What is the growth rate in this case?