Chapter 9: Nash equilibrium for monopolies and duopolies

We discuss here an application of Nash equilibrium in economics, the *Cournot's duopoly model*. This is a very classical problem which in fact predates modern game theory by more than a century.

Supply and demand: Imagine a number of companies which produce some item and sell it on the same market. The law of supply and demand is that the price of the product should decrease as the supply increases. We will use a simple equation to model this.

- P is the price of a single item.
- Q is the aggregate amount of items produced.

We assume that the price P at production level Q is given by

$$P(Q) = \begin{cases} P_0(1 - Q/Q_0) & \text{if} \quad Q \leq Q_0 \\ 0 & \text{if} \quad Q \geq Q_0 \end{cases} \tag{1}$$

The constant P_0 is the highest possible price and the constant Q_0 is the highest production level (determined by the size of the market).

Monopoly: As a warm-up let us imagine there is only one company serving the market. The company wants to maximize its profit. The profit, Π, is given by

$$\Pi = Q[P - c] = \text{Production level} \times \text{profit for each item} \tag{2}$$

where the constant c is the marginal cost, that is the price for the company to produce an item. Using the equation (1) we have

$$\Pi(Q) = QP_0\left(1 - \frac{Q}{Q_0}\right) - cQ\,. \tag{3}$$

To find the maximum we differentiate with respect to Q and find that the critical points are given by the solution of the equation

$$0 = \frac{d\Pi(Q)}{dQ} = P_0 - c - 2\frac{P_0}{Q_0}Q$$

with solution

$$Q^* = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)\,.$$

Note that Q^* is a maximum since the second derivative $\frac{d^2\Pi(Q)}{dQ^2} = -2\frac{P_0}{Q_0} < 0$.

To maximize its profit the company will choose the production level Q^*. The corresponding price is

$$P^* = P(Q^*) \;=\; P_0\left(1 - \frac{\frac{Q_0}{2}\left(1-\frac{c}{P_0}\right)}{Q_0}\right) \;=\; \frac{P_0+c}{2}$$

and the corresponding profit is

$$\begin{aligned}\Pi^* = Q^*[P^*-c] \;&=\; \frac{Q_0}{2}\left(1-\frac{c}{P_0}\right)\left(\frac{P_0+c}{2}-c\right)\\ &=\; \frac{Q_0P_0}{4}\left(1-\frac{c}{P_0}\right)^2 \qquad (4)\end{aligned}$$

So we obtain

Monopoly

$$Q^* = \frac{Q_0}{2}\left(1-\frac{c}{P_0}\right), \qquad P^* = \frac{P_0+c}{2}, \qquad \Pi^* = \frac{Q_0P_0}{4}\left(1-\frac{c}{P_0}\right)^2$$

Cartel: Suppose two companies R and C put the same product on the market. If they work as a *cartel* they will simply split the market in two and take half of it each. The price level will be unchanged and we have

Cartel

$$\widetilde{Q}_R = \widetilde{Q}_C = \frac{Q_0}{4}\left(1-\frac{c}{P_0}\right) \qquad \widetilde{P} = \frac{P_0+c}{2}, \qquad \widetilde{\Pi}_R = \widetilde{\Pi}_C = \frac{Q_0P_0}{8}\left(1-\frac{c}{P_0}\right)^2$$

Duopoly: To analyze a duopoly situation where two companies compete for the market, we realize that the competitive situation is nothing but a 2-player game.

- The *players* are the companies R and C.

- The *strategies* are, for both companies, R and C, to choose some production levels Q_R and Q_C.
- The *payoffs* Π_R and Π_C for R and C respectively are obtained by noting that the price P now depends on the aggregate production level $Q = Q_R + Q_C$:

$$\begin{aligned}\Pi_R(Q_R, Q_C) &= Q_R P_0 \left(1 - \frac{Q_R + Q_C}{Q_0}\right) - cQ_R \\ \Pi_C(Q_R, Q_C) &= Q_C P_0 \left(1 - \frac{Q_R + Q_C}{Q_0}\right) - cQ_C\end{aligned}$$

Recall that $\widehat{Q}_R$, $\widehat{Q}_C$ is a Nash equilibrium if $\widehat{Q}_R$ is a best response to $\widehat{Q}_C$ and $\widehat{Q}_C$ to $\widehat{Q}_R$. To find these value we first pick an arbitrary Q_C and compute the best response $\widehat{Q}_R$ to Q_C, that is we maximize the payoff $\Pi_R(Q_R, Q_C)$ with respect to Q_R. The best response is obtained by solving

$$0 = \frac{\partial \Pi_R(Q_R, Q_C)}{\partial Q_R} = P_0 - c - \frac{P_0}{Q_0} Q_C - 2\frac{P_0}{Q_0} Q_R \,,$$

and one finds

$$\widehat{Q}_R = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right) - \frac{Q_C}{2} \,.$$

By symmetry the best response $\widehat{Q}_C$ to Q_R is given by

$$\widehat{Q}_C = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right) - \frac{Q_R}{2}$$

If $\widehat{Q}_R$ and $\widehat{Q}_C$ are best responses to each other then they must satisfy the equations

$$\begin{aligned}\widehat{Q}_R &= \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right) - \frac{\widehat{Q}_C}{2} \\ \widehat{Q}_C &= \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right) - \frac{\widehat{Q}_R}{2}\end{aligned}$$

This is easy to solve and one finds the solution

$$\widehat{Q}_R = \widehat{Q}_C = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right) \,.$$

The corresponding price is

$$\widehat{P} = P(\widehat{Q}_R + \widehat{Q}_C) = P_0 \left(1 - \frac{\frac{2Q_0}{3}\left(1 - \frac{c}{P_0}\right)}{Q_0}\right) = \frac{1}{3}P_0 + \frac{2}{3}c$$

and the corresponding optimal payoffs are

$$\widehat{\Pi}_R = \widehat{Q}_R \left[\widehat{P} - c\right] = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right)\left(\frac{1}{3}P_0 + \frac{2}{3}c - c\right) = \frac{Q_0 P_0}{9}\left(1 - \frac{c}{P_0}\right)^2 \quad (5)$$

and the by symmetry the same solution for $\widehat{Q}_C$.

To summarize in a situation of duopoly where two firms compete for the market the analysis in terms of game theory gives the result

Duopoly

$$\widehat{Q}_R = \widehat{Q}_C = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right) \qquad \widehat{P} = \frac{1}{3}P_0 + \frac{2}{3}c\,, \qquad \widehat{\Pi}_R = \widehat{\Pi}_C = \frac{Q_0 P_0}{9}\left(1 - \frac{c}{P_0}\right)^2$$

Monopoly (or Cartel) vs Duopoly: By comparing the results we find that

- Compared to a monopoly, in a duopoly the price is lower $\widehat{P} < \widetilde{P}$ and the production level is higher $\widehat{Q} > \widetilde{Q}$, reflecting the availability of the products to a greater number of people.
- Since the production is higher in principle a greater number of people are employed to build the product.
- The aggregate profit for the companies is lower in a duopoly than in a monopoly.

As expected our model shows that competiton benefits the customers but diminishes the profits of the companies.