

## Chapter 7: Proportional Play and the Kelly Betting System

**Proportional Play and Kelly's criterion:** Investing in the stock market is, in effect, making a series of bets. Contrary to bets in a casino though, one would generally believe that the stock market is on average *rising*, so we are making a series of *superfair bets*.

In order to make things concrete we imagine that we invest in the stock market by buying (idealized) *option contracts*. With such a contract you may pay a certain amount  $b$  to have the possibility to sell a certain stock at a certain price at a given date. If the market stock price is below your option price you will buy it on the market and sell it immediately at a profit. if the stock price is above your option price then you do nothing and you lose the amount you paid for your option. We will make it even simpler by assuming your option cost \$1 (this your bet), and you make a profit  $\gamma$  with probability  $p$  and lose your bet with probability  $1 - p$ .

$$P(\text{Win } \$\gamma) = p \quad P(\text{Lose } \$1) = q = (1 - p)$$

If the game is superfair then we assume

$$\text{Expected gain } E[W] = \gamma p - q > 0.$$

Informed with our experience with the gambler's ruin we recall that in a subfair game the best strategy is *bold play*, that is invest everything at each step. But if the game is superfair then our probability to win a certain amount gets larger if we make small bets. But if we buy options that are sold on a fixed time window and making very small bets won't make you much money or only very slowly. To

**Proportional Play = Invest a proportion  $0 \leq f \leq 1$  of your fortune at each bet.**

Let us know computed your fortune after  $n$  bets. If you start with a fortune  $X_0$  then you bet  $fX_0$ . If you lose your fortune is  $X_0(1 - f)$  and if you win your fortune will be

$X_0(1 - f)$  (what you did not bet) plus  $fX_0$  (your original bet) plus  $\gamma fX_0$  (your gain). So we have

$$X_1 = \begin{cases} X_0 + \gamma fX_0 & \text{if you win} \\ X_0 - fX_0 & \text{if you lose} \end{cases} = \begin{cases} X_0(1 + \gamma f) & \text{with probability } p \\ X_0(1 - f) & \text{with probability } q = 1 - p \end{cases}$$

So at each step your fortune is *multiplied by a random factor* either  $1 + \gamma f$  or  $1 - f$ , so we define the random variable

$$Q = (1 + \gamma f) \text{ with probability } p \text{ and } Q = (1 - f) \text{ with probability } q.$$

To find our fortune at time  $N$  we take

$$Q_1, Q_2, \dots, Q_N \text{ independent identically distributed random variables}$$

all with the same distribution as  $Q$ . Then we have

$$\text{Fortune after } N \text{ bets is } X_N = Q_N Q_{N-1} \cdots Q_2 Q_1 X_0$$

Since we make a series of super fair bets we expect that our fortune to fluctuate but to increase exponentially

$$X_N \sim X_0 \exp \alpha N.$$

and so  $\alpha$  represents the rate at which our fortune increases. We can compute the long run value of  $\alpha$  by using the law of large numbers. Indeed we have

$$\alpha \sim \frac{1}{N} \log(X_N/X_0) = \frac{1}{N} [\log(Q_1) + \log(Q_2) + \cdots + \log(Q_N)]$$

and so by Chebyshev

$$\frac{1}{N} [\log(Q_1) + \log(Q_2) + \cdots + \log(Q_N)] \longrightarrow E[\log(Q)] \equiv \alpha$$

and we have

$$P(X_0 e^{N(\alpha-\epsilon)} \leq X_N \leq X_0 e^{N(\alpha+\epsilon)}) \geq 1 - \frac{\text{var}(\log(Q))}{N\epsilon^2}$$

To find the best asset allocation we seek to maximize  $\alpha = E[\log(Q)]$ .

**Optimal proportional play  $\iff$  maximize  $E[\log(Q)]$**

We have

$$E[\log(Q)] = p \log(1 + \gamma f) + q \log(1 - f)$$

Differentiating with respect to  $f$  we find

$$\begin{aligned} 0 &= \frac{d}{df} E[\log(Q)] \\ &= \frac{d}{df} p \log(1 + \gamma f) + q \log(1 - f) \\ &= \frac{p\gamma}{1 + \gamma f} - \frac{q}{1 - f} \end{aligned} \tag{1}$$

and so we have

$$\begin{aligned} p\gamma(1 - f) &= q(1 + \gamma f) \\ p\gamma - q &= f\gamma(p + q) = f\gamma \end{aligned}$$

Finally we obtain

**Optimal  $f$  :**     $f^* = \frac{p\gamma - q}{\gamma} = \frac{\text{Expected gain}}{\text{gain}}$     **Kelly's formula**

For example if your bet results in a payout of \$10 with probability  $1/4$  you should bet

$$f^* = \frac{10\frac{1}{4} - \frac{3}{4}}{10} = \frac{7}{40} = 0.175$$

of your fortune on each bet. Your fortune, on the long run, will grow at the rate of

$$\alpha = \frac{1}{4} \log\left(1 + 10\frac{7}{40}\right) + \frac{3}{4} \log\left(1 - \frac{7}{40}\right) = 0.047$$

**Horse races:** In the same spirit we can wonder what is the optimal strategy if we are betting on a horse race. We suppose there are  $m$  horses, and the betting odds are such that if we pay \$1 on on horse  $i$  then we earn  $\gamma_i$  if horse  $i$  wins. How should we bet to maximize our fortune in the long run? In order not to ever lose all our money it makes sense to split your money and bet on *every* horse. That is

**Proportional Play = Bet a proportion  $b_i$  of your fortune on horse  $i$ .**

In order to find the optimal proportion we assume that horse  $i$  has probability  $p_i$  to win. How do we find  $p_i$  is the interesting part of the problem. One possible way to estimate  $p_i$  is to trust the *wisdom of the crowd* and to use as  $p_i$  the proportion of players which bets on horse  $i$ . This is the idea behind *betting markets* where one assume that if many people bet real money on a certain outcome, then their collective knowledge will produce the "true" probability. This is a version of the *efficient market hypothesis* which asserts that stock always trade at their fair values. Or in the context of sport betting you may want to believe that a knowledgeable bettor will be able to find the right value of  $p_i$  by the virtue of his accumulated knowledge.

In any case, as in the Kelly betting system above we try to maximize the growth rate. If we bet  $b_i$  on horse  $i$  and wins we earn  $\gamma_i b_i$  so if we start with a fortune  $X$  after betting our fortune will be  $Q_b X$  where  $Q_b$  is a random variable with

$$P\{Q_b = b_i \gamma_i\} = p_i$$

and we need to find

$$\max_{b_i: \sum b_i = 1} E[\ln(Q_b)] = \max_{b_i: \sum b_i = 1} \sum_{i=1}^m p_i \ln(b_i \gamma_i)$$

This can be done using Lagrange multiplier. We set

$$F(b, \lambda) = \sum_{i=1}^m p_i \ln(b_i \gamma_i) + \lambda \sum_{i=1}^m b_i.$$

and differentiating with respect to  $b_i$  we find that the maximum should satisfy

$$p_i \frac{1}{b_i} + \lambda = 0 \quad \text{or} \quad b_i = -\frac{p_i}{\lambda}$$

Using that  $\sum_i b_i = 1$  we find  $\lambda = -1$  and thus

Optimal strategy is to bet a proportion  $b_i = p_i$  of your fortune.

**Asset allocation in the stock market:** A similar situation occurs in the stock market. We suppose there are  $m$  stocks and the  $i^{\text{th}}$  stock rate of return is described by the random variable  $Z_i$

$Z_i =$  ratio of the value of stock  $i$  tomorrow and the value of stock  $i$  today

that is  $Z_i = .98$  it means the stock  $i$  lost two percent today.

To invest our money we decide to invest a proportion  $b_i$  of our fortune in stock  $i$  with  $\sum_{i=1}^m b_i = 1$ . With this allocation our fortune after  $N$  days will be

$$X_N = Q_N \cdots Q_1 X_0$$

where  $Q_i$  are IID copies of  $Q$

$$Q = \sum_{i=1}^m b_i Z_i$$

and  $X_0$  is the initial fortune.

In order to maximize the rate of growth of the fortune, we need to find

$$\max_{b_i \geq 0, \sum b_i = 1} E[\log(\sum_{i=1}^m b_i Z_i)]$$