

Chapter 4: Gambler's ruin and bold play

Random walk and Gambler's ruin. Imagine a walker moving along a line. At every unit of time, he makes a step left or right of exactly one of unit. So we can think that his position is given by an integer $n \in \mathbf{Z}$. We assume the following probabilistic rule for the walker starting at n

$$\begin{aligned} \text{if in position } n \quad & \text{move to } n + 1 \text{ with probability } p \\ & \text{move to } n - 1 \text{ with probability } q \\ & \text{stay at } n \text{ with probability } r \end{aligned} \tag{1}$$

with

$$p + q + r = 1$$

Instead a walker on a line you can think of a gambler at a casino making bets of \$1 at certain game (say betting on red on roulette). He start with a fortune of \$ n . With probability p he doubles his bet, and the casino pays him \$1 so that he increase his fortune by \$1. With probability q he loses and his fortune decreases by \$1, and with probability r he gets his bet back and his fortune is unchanged.

As we have seen in previous lectures, in many such games the odds of winning are very close to 1 with p typically around .49. Using our second order difference equations we will show that even though the odds are only very slightly in favor of the casino, this is enough to ensure that in the long run, the casino will makes lots of money and the gambler not so much. We will also investigate what is the better strategy for a gambler, play small amounts of money (be cautious) or play big amounts of money (be bold). We shall see that being bold is the better strategy if odds are not in your favor (i.e. in casino), while if the odds are in your favor the better strategy is to play small amounts of money.

We say that the game is

$$\begin{aligned} \text{fair} & \quad \text{if } p = q \\ \text{subfair} & \quad \text{if } p < q \\ \text{superfair} & \quad \text{if } p > q \end{aligned}$$

The gambler's ruin equation: In order to make the previous problem precise we imagine the following situation.

- You starting fortune if \$ j .
- In every game you bet \$1.

- Your decide to play until you either loose it all (i.e., your fortune is 0) or you fortune reaches \$N and you then quit.
- What is the probability to win?

We denote by A_j the event that you win starting with \$j.

$$\begin{aligned} x(j) &= P(A_j) = \text{Probability to win starting with } j \\ &= \text{Probability to reach } N \text{ before reaching } 0 \text{ starting from } j \end{aligned} \quad (2)$$

To compute $x(j)$ we use the formula for conditional probability and condition on what happens at the first game, win, lose, or tie. For every game we have

$$P(\text{win}) = p, \quad P(\text{lose}) = q, \quad , P(\text{tie}) = r$$

We have

$$\begin{aligned} x(j) &= P(A_j) \\ &= P(A_j|\text{win})P(\text{win}) + P(A_j|\text{lose})P(\text{lose}) + P(A_j|\text{tie})P(\text{tie}) \\ &= x(j+1) \times p + x(j) \times q + x(j-1) \times r. \end{aligned} \quad (3)$$

To figure this out we realize that if we win the first game then our fortune is then $j+1$, and so $P(A_j|\text{win}) = P(A_{j+1}) = x(j+1)$, and similarly for the two other terms.

Note also that we have $x(0) = P(A_0) = 0$ since we have then nothing more to gamble and $x(N) = P(A_N) = 1$ since we have reached our goal and then stop playing. Using that $p+q+r=1$ we can rewrite this as the second order equation

Gambler's ruin $px(j+1) - (p+q)x(j) + qx(j-1) = 0, \quad x(0) = 0, x(N) = 1$

Trying $x(j) = \alpha^j$ we find the quadratic equation

$$p\alpha^2 - (p+q)\alpha + q = 0$$

with solutions

$$\alpha = \frac{p+q \pm \sqrt{(p+q)^2 - 4pq}}{2p} = \frac{p+q \pm \sqrt{p^2 + q^2 - 2pq}}{2p} = \frac{p+q \pm \sqrt{(p-q)^2}}{2p} = \begin{cases} 1 \\ q/p \end{cases}$$

If $p \neq q$ we have two solutions and and so the general solution is given by

$$x(j) = C_1 1^j + C_2 (q/p)^j$$

We will consider the case $p = q$ later. To determine the constants C_1 and C_2 we use that $x(0) = 0$ and $x(N) = 1$ and we find

$$0 = C_1 + C_2, \quad 1 = C_1 + C_2 (q/p)^N$$

which gives

$$C_1 = -C_2 = \frac{1}{1 - (q/p)^N}.$$

Finally we have then

Gambler's ruin probabilities $x(j) = \frac{1 - (q/p)^j}{1 - (q/p)^N} \quad p \neq q$

The Gambler's ruin for fair games: We briefly discuss the case of fair game $p = q$. In that case the equation for the ruin's probabilities x_j simplify to

$$x(j+1) - 2x(j) + x(j-1)$$

which gives the quadratic equation

$$\alpha^2 - 2\alpha + 1$$

with only one root $\alpha = 1$. So we have only one solution $x_j = C$. To find a second solution after some head-scratching and we try the solution $x_j = Dj$ and indeed we have

$$D(j+1) - 2Dj + D(j-1) = 0$$

so that the general solution is

$$x(j) = C_1 + C_2 j.$$

With $x(0) = 0$ and $x(N) = 1$ we find

Gambler's ruin probabilities $x_n = \frac{j}{N} \quad \text{if } p = q$

How bad does it get? To get an idea on how the gambler's ruin formula looks like let us take a *subfair game* with

$$q = .51, \quad p = 0.49, \quad r = 0,$$

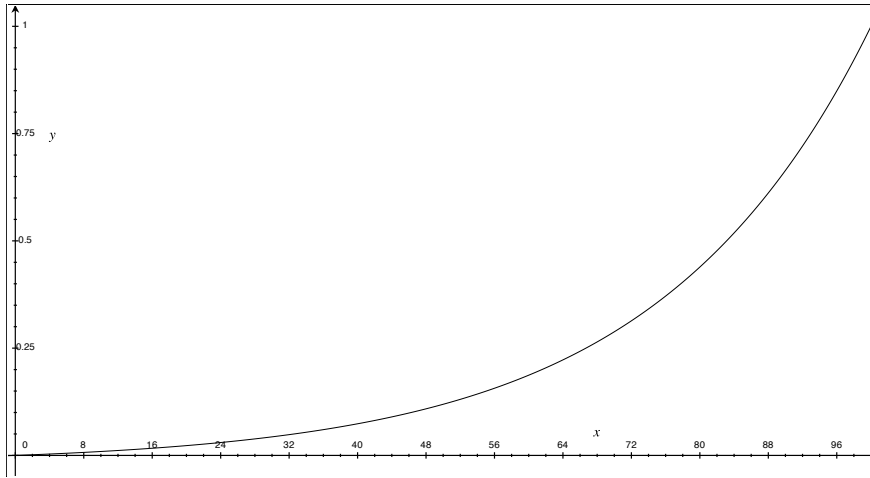


Figure 1: Gambler's ruin probabilities for $n = 100$, $p = 0.49$, $q = 0.51$, $r = 0$

see figure 1 and let us pick $N=100$. That is we start with a fortune of j and wish to reach a fortune of 100. We have for example.

$$x_{10} = 0.0091, \quad x_{50} = 0.1191, \quad x_{75} = 0.3560, \quad x_{83} = 0.4973.$$

That is starting with \$10 the probability to win \$100 before losing all your money is only about one in hundred. If you start half-way to your goal, that is with \$50, the probability to win is still not so great 11 in one hundred and you reach a fifty-fifty chance to win only if you start with \$83.

If we have a *superfair game* with

$$q = .49, \quad p = 0.51, \quad r = 0,$$

and pick $N=100$, then we have

$$x_{10} = 0.3358, \quad x_{50} = 0.8808, \quad x_{75} = 0.9679.$$

That is starting with \$10 the probability to win \$100 before losing all your money is only about one third. and if you start half-way to your goal, that is with \$50, the probability to win is already close to .9. This is quite a dramatic change from a subfair game!

Bold or cautious? Using the formula for the gambler's ruin we can investigate whether there is a better gambling strategy than betting \$1 repeatedly. For example if we start with \$10 and our goal is to reach \$100 we choose between

- Play in \$1 bets?

- Play in \$10 bets?

We find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } x_{10} = \frac{1 - (51/49)^{10}}{1 - (51/49)^{100}} = 0.0091$$

while if we bet \$10 at each game we use the same formula now with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } x_1 = \frac{1 - (51/49)}{1 - (51/49)^{10}} = 0.0829$$

that is your chance to win is about 8 in hundred, about nine time better than by playing in \$1 increments. Based on such arguments it seems clear the best strategy is to be bold if the odds of the game are not in your favor.

If on the contrary the odds are in your favor, even so slightly, say $q=.49$, and $p=.51$ then the opposite is true: you should play cautiously. For example with these probabilities and in the same situation as before, starting with \$ 10 and with a \$100 goal we find

$$\text{Probability to win \$100 in \$1 bets starting with \$10 is } x_{10} = \frac{1 - (49/51)^{10}}{1 - (49/51)^{100}} = 0.3358$$

while for the other case we use the same formula with $N = 10$ and $j = 1$ since we need to make a net total of 9 wins

$$\text{Probability to win \$100 in \$10 bets starting with \$10 is } x_{11} = \frac{1 - (49/51)}{1 - (49/51)^{10}} = 0.1189.$$

In summary we have

If the odds are in your favor be cautious but if the odds are against you be bold!

Two limiting cases: To get a better handle on the formula let us look at 2 limiting cases and we slightly rephrase the problem:

- We start at 0.
- We stop whenever we reach W (W stands for our desired gain) and when we reach $-L$ (L stands for how much money we are willing to lose).

Now $j \in \{-L, -L + 1, \dots, \dots, W - 1, W\}$. We have simply changed variables and so obtain

$$P(-L, W) \equiv P(\text{Reach } W \text{ before reaching } -L \text{ starting from } 0) = \frac{1 - (q/p)^L}{1 - (q/p)^{L+W}}.$$

We consider the two limiting cases where W and L go to ∞ .

- If L goes to infinity it means that the player is willing to lose an infinite amount of money, that is he has infinite resources and he is trying to reach a gain of W units. If $q/p < 1$ (superfair) then $(q/p)^L \rightarrow 0$ as $L \rightarrow \infty$ and so $P(-\infty, W) = 1$. On the other hand if $q/p > 1$ (subfair) the ratio is ∞/∞ and we factorize $(q/p)^L$ and find

$$\begin{aligned} \frac{1 - (q/p)^L}{1 - (q/p)^{L+W}} &= \frac{(q/p)^L \left((q/p)^{-L} - 1 \right)}{(q/p)^L \left((q/p)^{-L} - (q/p)^W \right)} \\ &= \frac{(q/p)^{-L} - 1}{(q/p)^{-L} - (q/p)^W} \rightarrow (p/q)^W \end{aligned}$$

So we find

$$\text{Prob that a gambler with unlimited resources ever gain } W = \begin{cases} 1 & \text{if } q < p \\ (p/q)^W & \text{if } p < q \end{cases}$$

This is *bad news*: even with infinite resources the probability to ever win a certain given amount in a casino is exponential small!

- If W goes to infinity it means that the player has no win limit and he will be playing either forever or until he loses his original fortune of L . If $q/p > 1$ then the denominator goes to infinity while if $q/p < 1$ it goes to 1. Thus we have

$$\text{Prob that a gambler with no win limit plays for ever} = \begin{cases} 1 - (q/p)^L & \text{if } q < p \\ 0 & \text{if } p < q \end{cases}$$

This is *bad news* again: in a casino the probability to play forever is 0.

How much is free money worth in casino? Imagine that one of your friend is an extremely rich owner of a casino and make you the following present. You can go his

casino and he will give you an *infinite credit*, provided that you play at a table of craps ($p = 244/495$, $q = 251/495$) which happens to have a house limit of \$15,000 (maximal bet allowed). You of course then decide to be bold and bet the maximum of \$15,000 every time. Based on your knowledge of the gambler's ruin formula, you fix yourself a goal of W (in units of \$15000) and deduce that you will reach W with probability $(p/q)^W$ and never reach W with probability $(1 - (p/q)^W)$. So on average you will make

$$15,000 W (p/q)^W$$

money with your free credit. Now you are free to fix W and the optimal choice W^* is obtained by maximization, that is

$$W^* = \operatorname{argmax} W (p/q)^W .$$

If we differentiate $f(W) = W (p/q)^W$ with respect to W we find that

$$\frac{d}{dW} W (p/q)^W = \frac{d}{dW} W e^{W \ln(p/q)} = e^{W \ln(p/q)} + W \ln(p/q) e^{W \ln(p/q)} = (p/q)^W + \ln(p/q) W (p/q)^W .$$

Setting the derivative equal to 0 gives

$$0 = 1 + \ln(p/q) W^* \quad \text{or} \quad W^* = \frac{1}{\ln(q/p)} = \frac{1}{\ln(251/244)} = 35.35$$

and the (average) amount of money you can extract out of this infinite credit line is a (paltry)

$$15,000 W^* (p/q)^{W^*} = 15,000 \frac{1}{\ln(q/p) e} = 15,000 \times 13.00 = 195,000 .$$

Bold play strategy. The gambler's ruin suggests that if the game is subfair one should bet one's entire fortune. Now if you are trying to reach a certain target there is no reason to bet more than necessary to reach that target, for example if your fortune is \$75 and you want to reach \$200 you will bet \$75 but if you want to reach \$100 you will bet only \$25, keeping some money left to try again if necessary. So we define the *bold play strategy* by

Bold play : Bet everything you can but no more than necessary

It turns out we can compute the probability to win using the bold play strategy by using a simple algorithm. First it is useful to normalize since there is no difference between wanting to reach \$100 with bets of \$1 or wanting to reach \$1000 with bets of \$10. So we normalize the target fortune to 1 and imagine we start with a fortune of z with $0 < z < 1$

$Q(z)$ = **probability to reach a fortune of 1 starting with a fortune of f and playing the bold strategy.**

Example: Let us compute $Q(1/2)$: in this case we bet everything to reach 1 and so $Q(1/2) = p$. For $Q(1/4)$ we first bet everything. If we lose, we lose everything while if we win we now have $1/2$. Thus we obtain

$$Q(1/4) = pQ(1/2) = p^2.$$

For $Q(3/4)$ we bet $1/4$ and we condition on the first bet. We have

$$Q(3/4) = Q(3/4|W)P(W) + Q(3/4|L)P(L) = p + Q(1/2)q = p(1 + q)$$

since if we start with $3/4$ and win, then we win, and if we lose then we are down to $1/2$

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Formula for bold play probabilities: We derive the basic equations for the bold play strategy. If your fortune z is less than $1/2$ then you bet z and ends up with fortune of $2z$ if you win and nothing if you loose. So by conditioning we find

$$Q(z) = Q(z|W)pP(w) + Q(z|L)P(L) = pQ(2z) + Q(0)q = pQ(2z)$$

On the other hand if your fortune z exceeds $1/2$ you will bet only $1 - z$ to reach 1. By conditioning you find

$$Q(z) = Q(z|W)pP(w) + Q(z|L)P(L) = Q(1)p + Q(z - (1 - z))q = p + qQ(2z - 1)$$

In summary we have

Bold play conditional probabilities

$$Q(z) = pQ(2z) \quad \text{if } z \leq 1/2$$

$$Q(z) = p + qQ(2z - 1) \quad \text{if } z \geq 1/2$$

$$Q(0) = 0, \quad Q(1) = 1$$

Example: dyadic rational. We say that z is a dyadic rational if z has the form $z = \frac{j}{2^l}$ for some l . Then $Q(z)$ can be computed recursively. For example we have already computed $Q(1/4)$, $Q(2/4)$ and $Q(3/4)$. We obtain using these values

$$\begin{aligned} Q(1/8) &= pQ(2/8) = p^3 \\ Q(3/8) &= pQ(6/8) = p^2(1+q) \\ Q(5/8) &= p + qQ(2/8) = p + qp^2 \\ Q(7/8) &= p + qQ(6/8) = p + qp(1+q) = p(1+q+q^2) \end{aligned}$$

and clearly we could now compute $Q(1/16)$, $Q(3/16)$, etc...

Example. We can compute $Q(z)$ if z is rational and if we are patient enough. Say we want to compute $Q(1/10)$. We try to find a closed system of equations.

$$\begin{aligned} Q(1/10) &= pQ(2/10) \\ Q(2/10) &= pQ(4/10) \\ Q(4/10) &= pQ(8/10) \\ Q(8/10) &= p + qQ(6/10) \\ Q(6/10) &= p + qQ(2/10) \end{aligned}$$

We can have an equation for $Q(2/10)$:

$$Q(2/10) = p^2Q(8/10) = p^3 + p^2qQ(6/10) = p^3(1+q) + p^2q^2Q(2/10)$$

so

$$Q(2/10) = \frac{p^3(1+q)}{1-p^2q^2}$$

and

$$Q(1/10) = \frac{p^4(1+q)}{1-p^2q^2}$$

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One can show that

Theorem: (Bold play is optimal) Consider a game where by you double your bet with probability p and lose your bet with probability q . If the game is subfair ($p < 1/2$) then the bold play strategy is optimal in the sense that it gives you the highest probability to reach a certain fortune by a series a bets of varying sizes.