Chapter 3: Linear Difference equations

In this chapter we discuss how to solve *linear difference equations* and give some applications. More applications are coming in next chapter.

First order homogeneous equation: Think of the time being discrete and taking integer values $n = 0, 1, 2, \cdots$ and $x(n)$ describing the state of some system at time n. We consider an equation of the form

First order homogeneous $x(n) = ax(n-1)$

where $x(n)$ is to be determined is a constant. This equation is called a *first order homogeneous* equation and it is easy to solve iteratively.

$$x(n) = ax(n-1) = a(ax(n-2)) \; = \; a^2x(n-2) \; = \; \cdots \; = \; a^n x(0).$$

So if we are given $x(0)$, i.e. the state of the system at time 0, then the state of the system at time n is given by $x(n) = a^n x(0)$, i.e. this is a model for exponential growth or decay.

To summarize

The general solution of $x(n) = ax(n-1)$ is $x(n) = Ca^n$

Interest rate: A bank account has a yearly interest rate of 5% compounded monthly. If you invest \$1000, how much money do you have after 5 years? Since the interest is paid monthly we set

$$x(n) = \text{ amount of money after } n \text{ months}$$

and since we get one twelfth of 5% every month we have

$$x(n) \; = \; \left(1 + \frac{.05}{12}\right) x(n-1) \; = \; \left(1 + \frac{1}{240}\right) x(n-1) \; = \; \left(\frac{241}{240}\right) x(n-1)$$

and so after 5 year we have with $x(0) = 1000$

$$x(60) \; = \; \left(\frac{241}{240}\right)^{60} 1000 \; = \; 1283.35$$

First order inhomogeneous equation: Let us consider an equation of the form

First order inhomogeneous $x(n) = ax(n-1) + b(n)$

where $b(n)$ is a *given* sequence and $x(n)$ is unknown. For example we may take

$$b(n) = b\,, \quad b(n) = 2n^2 + 3\,, \quad b(n) = b3^n\,.$$

This equation is called *inhomogeneous* because of the term $b(n)$. The following simple fact is useful to solve such equations

Linearity principle: *Suppose $x(n)$ is a solution of the homogeneous first order equation $x(n) = ax(n-1)$ and $y(n)$ is a solution of the inhomogeneous first order equation $y(n) = ay(n-1) + b(n)$.*

Then $z(n) = x(n) + y(n)$ is a solution of the inhomegenous equation $z(n) = az(n-1) + b(n)$. Indeed we have

$$\begin{aligned} z(n) &= x(n) + y(n) \\ &= ax(n-1) + ay(n-1) + b(n) \\ &= a[x(n-1) + y(n-1)] + b(n) \\ &= az(n-1) + b(n). \end{aligned}$$

To find the general solution of a first order homogeneous equation we need

- Find *one particular* solution of the inhomogeneous equation.
- Find the general solution of the homogeneous equation. This solution has a free constant in it which we then determine using for example the value of $x(0)$.
- The general solution of the inhomogeneous equation is the sum of the particular solution of the inhomogeneous equation and general solution of the homogeneous equation.

Example: Solve

$$x(n) = ax(n-1) + b$$

i.e., the inhomegeneous term is $b(n) = b$ is constant. We look for a particular solution, and after some head scratching we try $x(n) = D$ to be constant and find

$$D = aD + b\,, \qquad \text{or} \qquad D = \frac{b}{1-a}$$

The general solution is then

$$x(n) \;=\; Ca^n + \frac{b}{1-a}\,.$$

Example: Solve

$$2x(n) - x(n-1) = 2^n\,, \quad x(0) = 3$$

The solution of the homogenous equation $2x(n) - x(n-1)$ is $x(n) = C(1/2)^n$. To find a particular solution of the inhomogeneous problem we try an exponential function $x(n) = D2^n$ with a constant D to be determined. Plugging into the equation we find

$$2D2^n - D2^{n-1} = 2^n$$

or after dividing by 2^{n-1}

$$4D - D = 2 \text{ or } D = \frac{2}{3}\,.$$

So the general solution is

$$x(n) \;=\; C\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

and the initial condition gives $x(0) = 3 = C + \frac{2}{3}$ and so

$$x(n) \;=\; \frac{7}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}2^n\,.$$

More interest rate: A bank account gives an interest rate of 5% compounded monthly. If you invest invest initially \$1000, and add \$10 every month. How much money do you have after 5 years? Since the interest is paid monthly we set

$$x(n) = \text{ amount of money after } n \text{ months}$$

and we have the equation for $x(n)$

$$x(n) = \left(1 + \frac{.05}{12}\right)x(n-1) + 10 \;=\; \left(\frac{241}{240}\right)x(n-1) + 10$$

For the particular solution we try $x(n) = D$ and find

$$D = \frac{241}{240}D + 10$$

i.e., $D = -2400$. The general solution is then

$$x(n) = D\left(\frac{241}{240}\right)^n - 2400$$

and $x(0) = 1000$ gives

$$x(n) = 3400 \left(\frac{241}{240}\right)^n - 2400$$

and so $x(60) = 1963.41$

Second order homogeneous equation: We consider an equation where $x(n)$ depends on both $x(n-1)$ and $x(n-2)$:

Second order homogeneous $x(n) = ax(n-1) + bx(n-2)$.

It is easy to see that we are given both $x(0)$ and $x(1)$ we can then determine $x(2)$, $x(3)$, and so on.

Linearity Principle: *One verifies verify that if $x(n)$ and $y(n)$ are two solutions of the second order homogeneous equation, then $C_1 x(n) + C_2 y(n)$ is also a solution for any choice of constants C_1, C_2.*

To find the general solution we get inspired by the homogeneous first order equation and look for solutions of the form

$$x(n) = \alpha^n$$

If we plug this into the equation we find

$$\alpha^n = a\alpha^{n-1} + b\alpha^{n-2}$$

and dividing by α^{n-2} give

$$\alpha^2 - a\alpha + b = 0$$

We find (in general) two distinct roots α_1 and α_2 and the general solution has then the form

General solution $x(n) = C_1\alpha_1^n + C_2\alpha_2^n$

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Example: The **Fibonacci sequence** is given by

$$x(n) = x(n-1) + x(n-2), \quad x(0) = 0, x(1) = 1$$

that is every term of the sequence is the sum of the two preceding terms. It is given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \cdots$$

As we will see, the golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.61803398875$$

occurs in the Fibonacci sequence in the sense that for large n

$$\frac{x(n+1)}{x(n)} \approx \varphi \, .$$

For example $89/55 = 1.61818181818$, $144/89 = 1.61797752809$, $233/144 = 1.61805555556$, and so on... To see why it occurs we solve the second order difference equation: with $x(n) = \alpha^n$ we find

$$\alpha^2 - \alpha - 1 = 0$$

or

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

So the the general solution is

$$x(n) = C_1 \left(\frac{1+\sqrt{5}}{2} \right)^n +_2 \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

and with $x(0) = 0$ and $x(1) = 1$ we find

$$x(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] .$$

Since $|\frac{1-\sqrt{5}}{2}| < 1$ the second term is vanishingly small for large n so $x(n) \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$.
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Example: The **Fibonacci sequence and flipping coins**. The Fibonacci sequence shows up in many instances. In a probabilistic context it shows up in the following problem:

Determine the probability to flip a coin n times and have no successive heads.

To do this we need to *count* the number of sequences of heads (H) and tails (T) such that no successive heads occurs. So we set

$$f(n) \;=\; \text{number of sequences of } n \; H \text{ or } T \text{ without consecutive } H$$

and then we have

$$P\{\text{flip a coin } n \text{ times without consecutive heads}\} = \frac{f(n)}{2^n}$$

To find $f(n)$ we derive a recursive relation for it. Suppose we have a sequence of length n which ends up with a T. Then we can put in the first $n-1$ spots any sequence with no consecutive heads and this creates a sequence of length heads without consecutive heads. There are $f(n-1)$ such sequences. If the sequence of length n ends up with a H then the $n-1^{th}$ entry in the sequence needs to be T, one obtains then a sequence without consecutive heads if the first $n-2$ entries any sequence without consecutive heads. There are $f(n-2)$ such sequences and thus we found that

$$f(n) = f(n-1) + f(n-2)\,.$$

If $n = 1$ then we have $f(1) = 2$ and if $n = 2$ we have $f(2) = 3$ so that we obtain the Fibonacci sequence gain but shifted by two:

$$f(n) \;=\; x(n+2) \;=\; \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right]$$

As an example we find that the probability to flip a coin 15 times and have no successive heads is $\frac{x(17)}{2^{15}} = 0.0487$.

Second order inhomogeneous equation: We consider an equation of the form

$$\text{Second order homogeneous} \;\; x(n) = ax(n-1) + bx(n-2) + c(n)\,.$$

where $x(n)$ is unknown and $c(n)$ is a fixed sequence. As for first order equations we can solve such equations by

1. Solve the homogeneous equation $x(n) = ax(n-1) + bx(n-2)$.

2. Find a particular solution of the inhomogeneous equation.

3. Write the general solution as the sum of the particular inhomogeneous equation plus the general solution of the homogeneous equation.

Example: Find the general solution of the second order equation $3x(n) + 5x(n-1) - 2x(n-2) = 5$. For the homogeneous equation $3x(n) + 5x(n-1) - 2x(n-2) = 0$ let us try $x(n) = \alpha^n$ we obtain the quadratic equation

$$3\alpha^2 + 5\alpha - 2 = 0 \text{ or } \alpha = 1/3, -2$$

and so the general solution of the homogeneous equation is

$$x(n) = C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

For a particular equation $3x(n) + 5x(n-1) - 2x(n-2) = 5$ we try $x(n) = D$ and find

$$3D + 5D - 2D = 5$$

i.e. $D = 5/6$ and so the general solution is

$$x(n) = \frac{5}{6} + C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

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