Math 331.3 Spring 2017: Practice problems

Exercise 1 An object stretches a spring 6 inches in equilibrium.

1. Set up the equation of motion and find its general solution.
2. Find the displacement of the object for $t > 0$ if it is initially stretched upward 18 inches above equilibrium and given a upward velocity of 3 ft/s.
3. Write down the solution found in 2. in the form $R\cos(\omega_0 t - \phi)$ and determine the frequency, period, amplitude, and phase angle of the motion.

Exercise 2 A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is submitted to friction with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

Exercise 3 Consider the spring mass system

$$4\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 5y = 0\,,$$

where k is a parameter with $0 \le k < \infty$. As k varies describe the different types of the systems (damped, overdamped, undamped). Determine for which k a bifurcation occurs (this means that the system changes its type at that value..).

Exercise 4 Consider the spring mass

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + 2ky = 0\,,$$

where k is a parameter with $0 \le k < \infty$. As k varies describe the different types of the systems (damped, overdamped, undamped). Determine for which k a bifurcation occurs (this means that the system changes its type at that value..).

Exercise 5 Compute the general solution for the

$$
\begin{aligned}
(a) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{-4t} \\
(b) \quad & \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = e^{2t} \\
(c) \quad & \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = e^{t} + e^{-2t} \\
(d) \quad & \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 5\cos(3t) - t^2
\end{aligned}
$$

Exercise 6 Consider the forced spring mass system

$$\frac{dy^2}{dt^2} + 5y = 6\sin(\alpha t)\,.$$

where α is a parameter.

1. For which value of α does the system exhibit a *resonance*?
2. Find the general solution for the value of α found in (a)

Exercise 7 Consider the forced spring mass system

$$\frac{dy^2}{dt^2} + 4\frac{dy}{dt} + 7y = 6\sin(3t)\,.$$

1. Find the general solution.
2. Describe the behavior of the general solution as $t \to \infty$ (that is determine the *steady-state solution*) and graph a typical solution.
3. Compute the *amplitude and phase angle* of the steady state solution.

Exercise 8 Solve the initial value problem

$$\frac{dy^2}{dt^2} - 4\frac{dy}{dt} - 5y = 6\sin(3t)\,,\; y(0) = 2, y'(0) = -1\,.$$

Exercise 9 Consider the equation

$$\frac{dy^2}{dt^2} + 9y = 6\sin(3.1t)\,.$$

1. Determine the frequency of the *beating*.
2. Determine the frequency of the *rapid oscillations*.
3. Give a rough sketch of typical solution indicating clearly the results obtained in 1. and 2.

Remark: To answer this questions you do not need to compute the solution explicitly.

Exercise 10 Find the general solution of

$$(a)\;\frac{dy^2}{dt^2} + 16y = 3\sin(4t)\,.$$

$$(b)\;\frac{dy^2}{dt^2} + 16y = 5\cos(2t)$$

Exercise 11 Solve the initial value problem

$$(a)\;\frac{dy^2}{dt^2} + 16y = 3\sin(4t)\,,\; y(0) = 1, y'(0) = 0$$

$$(b)\;\frac{dy^2}{dt^2} + 16y = 5\cos(2t),\; y(0) = 2, y'(0) = 2$$

Exercise 12 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (c)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (d)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case of the five cases

1. Determine the type of the system, i.e., sink, source, saddle, center, spiral source, spiral sink, center, degenerate eigenvalues.
2. Draw the phase portrait of the system. If the eigenvalue are real you need to compute the eigenvectors and indicate them clearly on the phase portrait.
3. Draw a rough graph of a typical solution $y_1(t)$, $y_2(t)$. Note that you *do not* need to solve the system to do this! If the eigenvalues are complex indicate clearly in your graph the period of the oscillations.

Exercise 13 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where A is given by

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad (b)\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \quad (c)\begin{pmatrix} 1 & 1/4 \\ -17 & 2 \end{pmatrix} \quad (c)\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

In each case

1. Compute the *general solution* of $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$.
2. Solve the *initial value problem* $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$, $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercise 14 Consider the linear systems $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ where A is given by

$$(a)\begin{pmatrix} 1 & -\frac{9}{4} \\ \alpha & -4 \end{pmatrix} \quad (b)\begin{pmatrix} a & 3a \\ 1 & 2 \end{pmatrix}$$

and α is a parameter. Compute the eigenvalues as a function of α to d etermine the various types of the systems and the bifurcations as the parameter α varies.

Exercise 15 Compute the inverse Laplace transform of the following functions

$$(a)\ \frac{7}{s+2} \quad (b)\ \frac{e^{-8s}}{s(s+2)} \quad (c)\ \frac{e^{-5s}}{s^2+2s-5/4} \quad (d)\ \frac{1}{s^2+2s+2} \quad (e)\ \frac{e^{-2s}}{s^2+2s+2}$$

$$(g)\ \frac{2s-5}{s^2+2s+2} \quad (h)\ \frac{e^{-3s}}{(s^2+1)(s^2+4)} \quad (i)\frac{e^{-2s}}{(s-1)(s^2+4s+5)} \quad (j)\frac{e^{-5s}}{(s-1)(s^2+7s+10)}$$

Exercise 16 The function $f(t)$ is given by

$$f(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ 2 & \text{if} \quad 1 \le t < 3 \\ 1 & \text{if} \quad 3 \le t\,. \end{cases}$$

1. Compute the Laplace transform of $f(t)$. *Hint:* Write f as a combination of $u(t-a)$ for suitable a's.
2. Solve the equation $\dfrac{dy}{dt} + 3y = f(t)$.

Exercise 17 Use the Laplace transform method to solve the following initial value problems.

1. $\dfrac{dy}{dt} + 5y \;=\; 5u(t-2)\,,\; y(0) = -7\,.$ Make also a graph of the solutions.
2. $\dfrac{dy}{dt} + 4y \;=\; -3u(t-4)e^{2(t-4)}\,,\; y(0) = 2\,.$ What is $\lim_{t\to\infty} y(t)$?
3. $\dfrac{dy^2}{dt^2} + 4y \;=\; 2u(t-2)\cos(3(t-2)) \quad y(0) = 0,\; y'(0) = 1\,.$
4. $\dfrac{dy^2}{dt^2} + 4y \;=\; 3u(t-1)e^{-(t-1)} \quad y(0) = 0,\; y'(0) = 1\,.$ How does the solution behave for large t?
5. $\dfrac{dy^2}{dt^2} + 2\dfrac{dy}{dt} + 10y \;=\; u(t-4) \quad y(0) = 2,\; y'(0) = 0\,.$ What is $\lim_{t\to\infty} y(t)$? Make a graph of the solution.
6. $\dfrac{dy^2}{dt^2} + 5y \;=\; \delta(t-3) \quad y(0) = 2,\; y'(0) = 1\,.$ Make a graph of the solution.
7. $\dfrac{dy^2}{dt^2} + 4\dfrac{dy}{dt} + 7y \;=\; \delta(t-5) \quad y(0) = 6,\; y'(0) = -1\,.$ Make a graph of the solution.

Table of Laplace Transforms

$f(t)$	$\mathcal{L}(f(t))$		$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$			
t	$\frac{1}{s^2}$			Derivatives
t^2	$\frac{2}{s^3}$		y	$\mathcal{L}(y)$
t^n	$\frac{n!}{s^{n+1}}$		y'	$s\mathcal{L}(y) - y(0)$
e^{at}	$\frac{1}{s-a}$		y''	$s^2\mathcal{L}(y) - sy(0) - y'(0)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$			
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$			t-Shift
$\cosh(at)$	$\frac{s}{s^2-a^2}$		$f(t)$	$F(s)$
$\sinh(at)$	$\frac{a}{s^2-a^2}$		$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$			
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$			s-Shift
$\delta(t-a)$	e^{-as}		$f(t)$	$F(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$		$e^{at}f(t)$	$F(s-a)$