Math 331.2: Midterm practice problems

1 (a). Solve the initial value problem $\frac{dy}{dt} = y^2\,[t + \sin(t)]$ with $y(0) = 6$.

1(b). Solve the initial value problem $y' = \frac{3x^2 - 1}{3 + 2y}$ with $y(0) = 2$.

2. Mary initially deposits \$1000 in a savings account that pays interest at the rate of 5% per year (compounded continuously). She also arranges for \$25 per week to be deposited automatically into her account.

(a) Assume that weekly deposits can be approximated by continuous deposits. Write down an initial value problem for her account balance $S(t)$ over time (t measured in years).

(b) How long does she needs to save to buy a \$5000 car?

3. Consider the following equation for a certain population of squirrels given by $P(t)$ (t is measured in years).

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1)$$

(a) Find all the equilibrium points of the equations. Draw the phase line and determine the stability of each equilibrium points.

(b) Make a graph of the solutions with initial conditions $P(0) = 1/4$, $P(0) = 3/2$, and $P(0) = 3$.

(c) At a certain time the hunting of squirrels become permitted and the law allows that a certain percentage α of the squirrel population be eliminated every year. A new equation for the squirrel population is then

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1) - \alpha P$$

The IALS (International Association for the Liberation of Squirrels) asserts than no more than 10% of squirrels should be eliminated every year (i.e $\alpha = 0.1$), otherwise the population would go extinct. On the contrary the UHA (United Hunters of America) asserts that it is safe to hunt half of the squirrel population every year (i.e. $\alpha = 0.5$). Analyze the systems as α varies and determine who is right from the IALS or the UHA.

4. (a) Solve the initial value problem $\frac{dy}{dt} = -3y - 5e^{2t} + 3\,,\; y(0) = 4$.

4. (b) Solve the initial value problem $\frac{dy}{dt} = -3y/t - 2 - t^{-4}\,,\; y(1) = 4$.

4. (c). Solve the initial value problem $\frac{dy}{dt} = -e^t/y\,,\; y(0) = -2$.

5. The isotope Iodine 131 is used for certain medical treatment. It has a half-life of 8.04 days. An hospital receive a shipment of 500mg of Iodine 131. How much will be left after 20 days.

6. Consider the differential equation $\frac{dy}{dt} = y^2 - 4y + 2\,.$

(a) Find all the equilibrium points, determine their stability.

(b) Draw the phase line of the system.

(c) Sketch the solutions of the equation with the initial conditions (a) y(0)= 1, (b) y(0)=-5, and (c) y(0)=6. Indicate clearly the behavior for $t \to \infty$ and $t \to -\infty$.

7. A home buyer can afford to spend no more than \$1000 per month on mortgage payments. Suppose that the interest rate is 5% (per year) and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow.

(b) Determine the total interest paid during the term of the mortgage

8.(a) Solve the initial value problem $\frac{dy}{dt} = 6y + e^t + 2e^{2t}\,, \quad y(0) = 3\,.$

8. (b) Solve the initial value problem $\frac{dy}{dt} = \frac{1+y^2}{y}\,, \quad y(0) = -2\,.$

8. (c) Solve the initial value problem $\frac{dy}{dt} = 1 + y^2\,, \quad y(0) = -2\,.$

8. (d) Find the general solution $x^2 + y^2 + 2xyy' = 0$.

8. (e) Find the general solution $xy^2 + 2xyy' = 0$.

8. (f) Solve the initial value problem $\frac{dy}{dx} = 2y - y^4\,, \quad y(0) = 1\,.$

9 . After a drought a water reservoir of capacity 200 liters is only half-full with pure water. The rate at which water is taken from the reservoir is 1 liters per hour. To replenish the reservoir it is decided to pump water into the half empty reservoir at a rate of 2 liters per hour until the reservoir is full. The water pumped into the reservoir is mistakenly polluted with salt with a concentration of 25g per liter. Write down the equation for the amount of salt in the pound at time t. How much salt will there be in the reservoir when the reservoir is full?

10 Consider the differential equation $\frac{dy}{dt} = 3y^3 - 12y$ (a) Find the equilibrium points, draw the phase line, and identify the stability of the equilibrium points.
(b) Sketch the solutions with initial conditions $y(0) = 2$, $y(0) = -1$, $y(0) = -3$.

11. Consider the equation $\frac{dy}{dt} = y^2 + \alpha y\,.$ where $-\infty < \alpha < \alpha$ is a parameter.

(a) Determine the equilibrium points.

(b) Draw the phase line of the system, you should distinguish between two cases, depending on the value of α. What is the value of α at which the bifurcation occur? In both case determine the stability of the equilibrium points

12. Solve the initial value problem $\frac{dy}{dt} = yt + 2t\,,\ y(3) = 2\,.$

13. Newton's law of cooling for the temperature of a bady $u(t)$ is $\frac{du}{dt} = -k(u - T)$ where T is the ambient temperature and $k > 0$ a constant.

In a 70^o room a 50^o beer bottle is discovered on a kitchen counter and then minutes later the bottle is 60^o. If the refrigerator is kept at 40^o how long has the bottle of beer been sitting on the counter?

14. Find the solution of the initial value problem $2y'' - 3y' + y = 0\,,\ y(0) = 2, y'(0) = \frac{1}{2}$. Sketch a graph of the solution. Determine the maximum value of the solution. Find the point where the solution is 0.

15. Solve the initial value problems and *sketch a graph of the solution.*

(a) $y'' - y' - 2y = 0, y(0) = -1, y'(0) = 2$

(b) $y'' + 25y = 0, y(0) = 1, y'(0) = -1$

(c) $y'' + 2y' + 3y = 0, y(0) = 1, y'(0) = 0$

(d) $y'' - 4y' + 13y = 0, y(0) = 4, y'(0) = 3$

(e) $y'' - 4y' + 4y = 0, y(0) = 2, y'(0) = -1$

16. Consider the initial value problems $y'' + y' - 2y = 0,\ y(0) = 2, y'(0) = \beta$.

(a) For which value of β the solution satisfies $\lim_{x\to\infty} y(x) = 0$?
(b) For which values of β does the solution never hit 0?

17. Find the general solution of $x^2y'' - 3xy' + \frac{7}{4}y = 0$.

18. Suppose a mass of 10 kg stretches a spring by .5m (recall$g = 9.8m/sec^2$) and we assume there is no damping.

1. What is the natural frequency of the motion?
2. Assume that the mass is initially pulled down by 0.2m and given an upward velocity of 1m/sec. What is the amplitude of the motion? What is the phase shift?

19. Consider a mass-spring system given by the equation $2y'' + 4y' + 6.5y = 0$. Is it overdamped, underdamped, critically damped? Sketch a typical solution of the system (without solving the equation).