Solve problem 1 and 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____.

1. (16 points) a) Show that the Laurent series of \( \frac{1}{\sin(z)} \), centered at 0, has the form

\[
\frac{1}{\sin(z)} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots \text{ terms of order at least five.} \tag{1}
\]

(You can use equality (1) in the subsequent parts, even if you do not derive it).

b) Find the principal part at \( z = 0 \) of the function \( f(z) = \frac{1 - z}{z^5 \cdot \sin(z)} \).
c) Find all the singularities of \( f(z) \) (given in part b) in the disk \( \{|z| < 4\} \) and determine their type (isolated, removable, pole of what order, essential).

d) Find the residue at each isolated singularity in \( D \).
2. (12 points) a) Compute \( \sin\left(\frac{\pi}{4} + i \ln(3)\right) \). Simplify your answer as much as possible.

b) Find all solutions of the equation \( \cos(z) = i \).
3. (12 points) Compute the integral \( \int_C \frac{\sin(z) + 1}{e^{3z} - e^z} \, dz \), where \( C \) is the circle \( \{|z| = 1\} \) traversed counterclockwise.
4. (12 points) a) Find the Taylor series of the function \( f(z) = \frac{z + 1}{z - 1} \) centered at 0 and determine its radius of convergence. Justify your answer.

b) Find the Laurent series of the function \( f(z) \), given in part a), valid in the domain \( |z| > 1 \).
5. (12 points) a) Use the definition of contour integrals, in order to prove the equality

\[ \int_C e^\bar{z} \, dz = \int_C e^{4/z} \, dz, \] (2)

where \( C \) is the circle \( \{ |z| = 2 \} \), traversed counterclockwise.

*Caution: The exponent of the integrand, on the left hand side, is the complex conjugate \( \bar{z} \) of \( z \).*

b) Find the Laurent series of \( e^{4/z} \) centered at zero and classify the type of singularity at \( z = 0 \).

c) Use the equality (2) in order to evaluate the integral \( \int_C e^\bar{z} \, dz \).
6. (12 points) Evaluate the integral

\[ \int_{0}^{2\pi} \frac{d\theta}{5 + 4 \sin(\theta)}. \]
7. (12 points) Let $C_A$ be the straight line segment from $A + iA$ to $-A + iA$, where $A$ is a positive real number. Prove the inequality

$$\left| \int_{C_A} \frac{e^{iz}}{z^2 + 1} \, dz \right| \leq \frac{2Ae^{-A}}{A^2 - 1}.$$
8. (12 points) Determine whether the following statements are true or false. Justify your answers!

a) If \( f(z) \) and \( g(z) \) are analytic at a point \( z_0 \) and \( g(z_0) = g'(z_0) = 0 \), but both \( f(z_0) \) and \( g''(z_0) \) are non-zero, then

\[
\text{Res}_{z=z_0} \left( \frac{f}{g} \right) = 0.
\]

b) There exists an entire non-constant function \( f(z) \) satisfying the inequality

\[
|f(z)| \leq |z| e^{-|z|}.
\]
c) If $C$ is a simple closed contour, and $z_0$ does not belong to the domain $D$ bounded by $C$, then there is a single valued branch of $\log(z - z_0)$, defined for all $z$ in $D$.


d) There exists an entire function, whose real part is $xe^y$. 
9. (12 points) Evaluate the improper integral

\[ \int_{0}^{\infty} \frac{dx}{x^4 + 1} \]