1) **(15 points)** Given that the first few terms of the Laurent series for the function \( \cot z \) around \( z = 0 \) are:

\[
\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots
\]

(i) Find the principal part at \( z = 0 \) of the function \( f(z) = \frac{(1 + z) \cot z}{z^4} \).

(ii) Find all the singularities of \( f(z) \) in the disk \( D = \{ |z| < 5 \} \). Determine the nature of each singularity (isolated, removable, pole of what order, essential).

(iii) Find the residue at each isolated singularity in \( D \).
2) (10 points) Compute: $\int_C \frac{\cos z}{e^{iz} - 1} \, dz$ where $C$ is the circle $\{|z| = 2\}$ (traversed counterclockwise).

3) (10 points) Compute: $\int_C (e^{\sin z} + \bar{z}) \, dz$, where $C$ is the circle $\{|z| = 2\}$ (traversed counterclockwise).
4) (10 points) Compute: \[ \int_{0}^{2\pi} \frac{d\theta}{2 + \cos \theta} \]
5) (15 points) Compute: \[ \int_0^\infty \frac{x^2}{1 + x^6} \, dx \]
6) (10 points) (a) Find the Laurent series of the function \( f(z) = \frac{\log z}{z - i} \) around the point \( z_0 = i \).

(b) Find the Taylor series of the function \( f(z) = \frac{1}{z^2 - 3z + 2} \) around the point \( z_0 = 0 \).
(a) The limit $\lim_{z \to 0} \frac{e^{\frac{z}{z}} - 1}{z}$ exists and is equal to 1.

(b) There is a function $f(z)$, analytic in the disk $D = \{|z| < 1\}$, such that

$$|f(z)|^2 = 4 - |z|^2 \quad \text{for all} \quad z \in D$$

(c) If $f(z)$ has an isolated singularity at $z_0$ and $\text{Res}_{z_0}(f) = 0$, then $z_0$ is a removable singularity.
8) **(5 points)** Compute \( \cos \left( \frac{\pi}{2} - i \ln 2 \right) \). Simplify your answer as much as possible.

9) **(5 points)** Prove that \(| \int_\mathcal{C} e^{iz^2} \, dz | < 5\), where \( \mathcal{C} \) is the piece of the circle \(|z| = 2\) going from 2 to \(2i\) counter-clockwise.
10) (5 points) Find an entire function $f(z)$ such that $\Re(f) = 4x^3 y - 4xy^3 - y$. 