DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH. 421 – FINAL EXAM

12/16/99 **NAME:**__________________________

1) (15 points) Given that the first few terms of the Laurent series for the function $\cot z$ around $z = 0$ are:

$$\cot z \;=\; \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots$$

(i) Find the principal part at $z = 0$ of the function $f(z) \;=\; \dfrac{(1+z)\cot z}{z^4}$.

(ii) Find all the singularities of $f(z)$ in the disk $D = \{|z| < 5\}$. Determine the nature of each singularity (isolated, removable, pole of what order, essential).

(iii) Find the residue at each isolated singularity in D.

2) (10 points) Compute: $\int_C \frac{\cos z}{e^{iz}-1}\,dz$ where C is the circle $\{|z|=2\}$ (traversed counter-clockwise).

3) (10 points) Compute: $\int_C (e^{\sin z}+\bar{z})dz$, where C is the circle $\{|z|=2\}$ (traversed counter-clockwise).

4) (10 points) Compute: $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

5) (15 points) Compute: $\int_0^{\infty} \frac{x^2}{1+x^6} dx$

6) (10 points) (a) Find the Laurent series of the function $f(z) = \dfrac{\operatorname{Log} z}{z - i}$ around the point $z_0 = i$.

(b) Find the Taylor series of the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$ around the point $z_0 = 0$.

7) (15 points) Determine whether the following statements are true or false. Justify your answers.

(a) The limit $\lim_{z \to 0} \frac{e^{\bar{z}} - 1}{z}$ exists and is equal to 1.

(b) There is a function $f(z)$, analytic in the disk $D = \{|z| < 1\}$, such that

$$|f(z)|^2 \; = \; 4 - |z|^2 \quad \text{for all} \quad z \in D$$

(c) If $f(z)$ has an isolated singularity at z_0 and $\mathrm{Res}_{z_0}(f) = 0$, then z_0 is a removable singularity.

8) (5 points) Compute $\cos\left(\frac{\pi}{2} - i\ln 2\right)$. Simplify your answer as much as possible.

9) (5 points) Prove that $\left|\int_C e^{iz^2}\, dz\right| < 5$, where C is the piece of the circle $|z| = 2$ going from 2 to $2i$ counter-clockwise.

10) (5 points) Find an entire function $f(z)$ such that $\text{Re}(f) = 4x^3y - 4xy^3 - y$.