Math 331, Fall 2017: Midterm practice problems

1. Solve the initial value problem $\frac{dy}{dt} = y^2 [t + \cos(t)]$ with $y(0) = 6$.

2. Solve the initial value problem $y' = \frac{3x \sin(x^2) - 1}{3 + 2y}$ with $y(0) = 2$.

3. Mary initially deposits \$1000 in a savings account that pays interest at the rate of 5% per year (compounded continuously). She also arranges for \$25 per week to be deposited automatically into her account.

   (a) Assume that weekly deposits can be approximated by continuous deposits. Write down an initial value problem for her account balance $S(t)$ over time (t measured in years).

   (b) How long does she needs to save to buy a \$5000 car?

4. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

5. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

6. At what yearly rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

7. Newtons law of cooling states that if an object with temperature T(t) at time t is in a medium with temperature T_m the rate of change of T at time t is proportional to $T(t) - T_m$, thus T satisfies a differential equation of the form

$$T' = -k(T - T_m)$$

   Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. Well call k the temperature decay constant of the medium.

   (a) A thermometer is moved from a room where the temperature is 70F to a freezer where the tem- perature is 12F . After 30 seconds the thermometer reads 40F. What does it read after 2 minutes?

   (b) An object is placed in a room where the temperature is 20C. The temperature of the object drops by 5C in 4 minutes and by 7C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

8. Consider the following equation for a certain population of squirrels given by $P(t)$ (t is measured in years).

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P - 1)$$

   (a) Find all the equilibrium points of the equations. Draw the phase line and determine the stability of each equilibrium points.

(b) Make a graph of the solutions with initial conditions $P(0) = 1/4$, $P(0) = 3/2$, and $P(0) = 3$.

(c) At a certain time the hunting of squirrels become permitted and the law allows that a certain percentage α of the squirrel population be eliminated every year. A new equation for the squirrel population is then

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{2}\right)(P-1) - \alpha P$$

The IALS (International Association for the Liberation of Squirrels) asserts than no more than 10% of squirrels should be eliminated every year (i.e $\alpha = 0.1$), otherwise the population would go extinct. On the contrary the UHA (United Hunters of America) asserts that it is safe to hunt half of the squirrel population every year (i.e. $\alpha = 0.5$). Analyze the systems as α varies and determine who is right from the IALS or the UHA.

9. Solve the initial value problem $\frac{dy}{dt} = -3y/t - 2 - t^{-4}\,,\ y(1) = 4$.

10. Solve the initial value problem $\frac{dy}{dt} = -e^t/y\,,\ y(0) = -2$.

11. A home buyer can afford to spend no more than \$1000 per month on mortgage payments. Suppose that the interest rate is 5% (per year) and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

    (a) Determine the maximum amount that this buyer can afford to borrow.

    (b) Determine the total interest paid during the term of the mortgage

12. Solve the initial value problem $\frac{dy}{dt} = yt + 2t\,,\ y(3) = 2\,.$

13. Solve the initial value problem $\frac{dy}{dt} = 9y + e^{-3t}\,, \quad y(0) = 3\,.$

14. Solve the initial value problem $\frac{dy}{dt} = 6y + 2t - 4\,, \quad y(0) = 3\,.$

15. Solve the initial value problem $\frac{dy}{dt} = \frac{1+y^2}{ye^{-x}}\,, \quad y(0) = -2\,.$

16. Solve the initial value problem $\frac{dy}{dx} = 1 + y^2\,, \quad y(0) = -2\,.$

17. Solve the initial value problem $\frac{dy}{dx} = y(y+2)\,, \quad y(0) = 1\,.$

18. Find the general solution of $x^2 + y^2 + 2xyy' = 0$.

19. Solve the initial value problem $(\sin(x) - y\sin(x) - 2\cos x) + \cos(x)y' = 0, \quad y(0) = -1.$

20. Find the general solution of $xy^2 + 2xyy' = 0$.

21. Find the general solution of $y' = \frac{y}{x} + e^{-y/x}$ *Hint: This one calls for a change of variable.*

22. Find the general solution of $\frac{dy}{dx} - y = xy^2$. *Hint: This one calls for a change of variable.*

23. Find all functions $M(x, y)$ such that the equation $M(x, y)dx + (x^2 - y^2)dy = 0$ is exact.

24. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with 1/2 pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

25. Consider the differential equation $\frac{dy}{dt} = 3y^3 - 12y$

    (a) Find the equilibrium points, draw the phase line, and identify the stability of the equilibrium points.

    (b) Sketch the solutions with initial conditions $y(0) = 2$, $y(0) = -1$.

26. Solve the initial value problems and *sketch a graph of the solution.*

    (a) $2y'' - 3y' + y = 0\,,\ \ y(0) = 2, y'(0) = \frac{1}{2}.$

    (b) $y'' - y' - 2y = 0, y(0) = -1, y'(0) = 2$

    (c) $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 0$

27. Consider the initial value problems $y'' + y' - 2y = 0$, $y(0) = 2, y'(0) = \beta$.

    (a) For which value of β the solution satisfies $\lim_{t\to\infty} y(t) = 0$?

    (b) For which values of β does the solution never hit 0?