SOLUTIONS FOR HOMEWORK SECTION 6.4 AND 6.5

Problem 1: For each of the following functions do the following: (i) Write the function as a piecewise function and sketch its graph, (ii) Write the function as a combination of terms of the form $u_a(t)k(t-a)$ and compute the Laplace transform

(a) $f(t) = t(1 - u_1(t)) + e^t(u_1(t) - u_2(t))$

(b) $h(t) = \sin(2t) + u_\pi(t)(t/\pi - \sin(2t)) + u_{2\pi}(t)(2\pi - t)/\pi$

(c) $g(t) = u_0(t) + \sum_{k=1}^{5}(-1)^k u_k(t)$

Solution:

(a)

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ e^t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

The graph is sketched in figure 1.

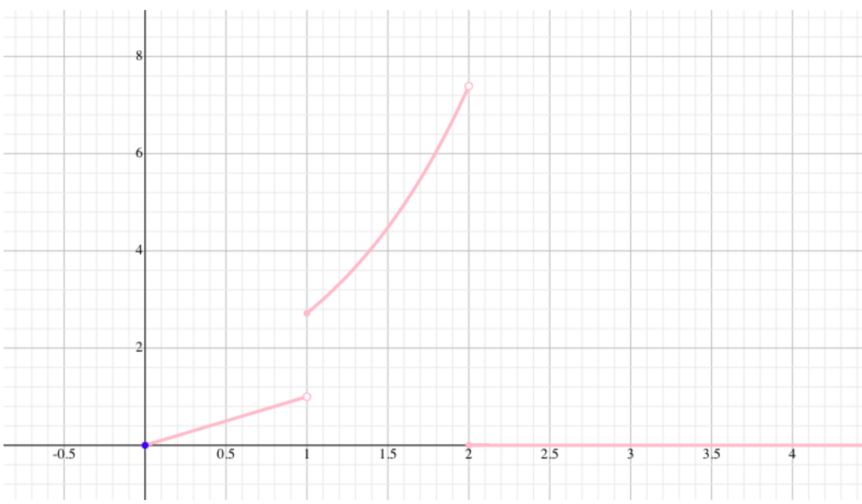

FIGURE 1. graph of f(t)

To find the Laplace transform of $f(t)$, rewrite $f(t)$ as

$$f(t) = t + (e^t - t)u_1(t) - e^t u_2(t)$$

$$\mathcal{L}\{f\} = \mathcal{L}\{t\} + \mathcal{L}\{e^t u_1(t)\} - \mathcal{L}\{tu_1(t)\} - \mathcal{L}\{e^t u_2(t)\}$$

Apply t-shifting theorem (Theorem 6.3.1 in the textbook) or s-shifting theorem if needed, finding the Laplace transform for each term:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{e^t u_1(t)\} = e\mathcal{L}\{e^{(t-1)}u_1(t)\} = e \cdot e^{-s}\mathcal{L}\{e^t\} = \frac{e^{(1-s)}}{s-1}$$

$$\mathcal{L}\{tu_1(t)\} = \mathcal{L}\{(t-1)u_1(t)\} + \mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

$$\mathcal{L}\{e^t u_2(t)\} = e^2\mathcal{L}\{e^{(t-2)}u_2(t)\} = e^2 \cdot e^{-2s}\mathcal{L}\{e^t\} = \frac{e^{(2-2s)}}{s-1}$$

Combining all terms yields

$$\mathcal{L}\{f\} = \frac{1}{s^2} + \frac{e^{(1-s)}}{s-1} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{(2-2s)}}{s-1}$$

(b)

$$h(t) = \begin{cases} sin(2t) & 0 \leq t \leq \pi \\ t/\pi & \pi \leq t < 2\pi \\ 2 & \pi \geq 2 \end{cases}$$

The graph is sketched in figure 2.

FIGURE 2. graph of h(t)

To find the Laplace transform of $h(t)$,

$$\mathcal{L}\{h\} = \mathcal{L}\{sin(2t)\} + \mathcal{L}\{u_\pi(t)(t/\pi - sin(2t))\} + \mathcal{L}\{u_{2\pi}(t)(2\pi - t)/\pi\}$$

Apply t-shifting theorem if needed, and find the Laplace transform of each term:

$$\mathcal{L}\{sin(2t)\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{u_\pi(t)(t/\pi - sin(2t))\} = e^{-\pi s}\mathcal{L}\{(\frac{t+\pi}{\pi} - sin(2(t+\pi)))\} = e^{-\pi s}(\frac{1}{\pi s^2} + \frac{1}{s} - \frac{2}{s^2+4})$$

$$\mathcal{L}\{u_{2\pi}(t)(2\pi - t)/\pi\} = e^{-2\pi s}\mathcal{L}\{\frac{2\pi - (t+2\pi)}{\pi}\} = -e^{-2\pi s}\frac{1}{\pi s^2}$$

Combining all terms yields

$$\mathcal{L}\{h\} = \frac{2}{s^2+4} + e^{-\pi s}(\frac{1}{\pi s^2} + \frac{1}{s} - \frac{2}{s^2+4}) - e^{-2\pi s}\frac{1}{\pi s^2}$$

(c) Here $u_0(t) = 1$ since we are only interested in the domain $t \geq 0$.

$$g(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \\ 1 & 2 \leq t < 3 \\ 0 & 3 \leq t < 4 \\ 1 & 4 \leq t < 5 \\ 0 & t \geq 5 \end{cases}$$

The graph is sketched in figure 3.

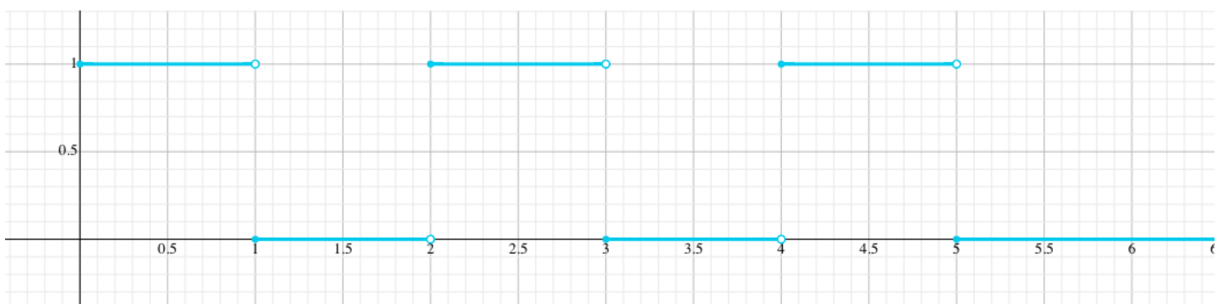

FIGURE 3. graph of g(t)

To find the Laplace transform of $g(t)$,

$$\mathcal{L}\{g\} = \mathcal{L}\{1\} + \sum_{k=1}^{5}\mathcal{L}\{(-1)^k u_k(t)\}$$
$$= \frac{1}{s} + \sum_{k=1}^{5}(-1)^k\frac{e^{-ks}}{s}$$
$$= \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s}$$

Problem 2: Solve the initial value problem

$$y' + 6y = g(t) \quad \text{where} \quad g(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 12 & \text{if } 1 \leq t < 7 \\ 0 & \text{if } 0 \leq t \end{cases}$$

with initial value $y(0) = 4$

Solution:

Use step function to represent $g(t)$ as

$$g(t) = 12(u_1(t) - u_7(t))$$

Take the Laplace transform of the differential equation and plug in initial value to get

$$sY(s) - 4 + 6Y(s) = 12(\frac{e^{-s}}{s} - \frac{e^{-7s}}{s})$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{12e^{-s}}{s(s+6)} - \frac{12e^{-7s}}{s(s+6)} + \frac{4}{s+6}$$

Let $H(s) = \frac{1}{s(s+6)}$, by partial fraction, we can rewrite $H(s)$ as

$$H(s) = \frac{1}{6}(\frac{1}{s} - \frac{1}{s+6})$$

The inverse Laplace transform of $H(s)$ is

$$h(t) = \mathcal{L}^{-1}\{H\} = \frac{1}{6}(1 - e^{-6t})$$

Hence, by using t-shifting theorem if necessary, we can find the solution

$$y(t) = \mathcal{L}\{Y\} = 12u_1(t)h(t-1) - 12u_7(t)h(t-7) + 4e^{-6t}$$

Problem 3: Solve the initial value problem

$$y'' + y = g(t) \quad \text{where} \quad g(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t \end{cases}$$

and with $y(0) = 0$ and $y'(0) = 0$.

Solution:

Firstly we can rewrite $g(t)$ as

$$g(t) = t(1 - u_1(t))$$

To find the Laplace transform of $g(t)$, we need use t-shifting theorem

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)t\} = \frac{1}{s} - e^{-s}\mathcal{L}\{t+1\} = \frac{1}{s} - e^{-s}(\frac{1}{s^2} + \frac{1}{s})$$

Then take the Laplace transform of the differential equation and plug in initial values to get

$$s^2Y(s) + Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)}$$

Let

$$H(s) = \frac{1}{s(s^2+1)} \quad \text{and} \quad F(s) = \frac{1}{s^2(s^2+1)}$$

We could use the method of undetermined coefficients method to find the partial fraction for $H(s)$:

$$H(s) = \frac{1}{s} - \frac{s}{s^2+1} \quad \text{and} \quad F(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

The inverse Laplace transform of them are

$$h(t) = 1 - \cos(t) \quad \text{and} \quad f(t) = t - \sin(t)$$

By applying t-shifting theorem if necessary, the inverse Laplace transform of $Y(s)$ is:

$$\begin{aligned} y(t) &= \mathcal{L}\{H(s)\} - \mathcal{L}\{e^{-s}H(s)\} - \mathcal{L}\{e^{-s}F(s)\} \\ &= h(t) - u_1(t)h(t-1) - u_1(t)f(t-1) \end{aligned}$$

Problem 4: Consider the mass-spring system described by the initial value problem

$$y'' + 4y = \sin t + u_{\pi/2}(t)\cos t; \;\; y(0) = 0, \; y'(0) = 0.$$

Find the solution of the initial value problem.

Hint: Rewrite $\cos t$ *using a trigonometric identity.*

Solution:

First need to write $u_{\pi/2}(t)\cos t$ in the form $u_c(t)f(t-c)$. To do this use the trig identities $\cos\theta = \sin(\pi/2 - \theta)$ and $\sin(-\theta) = -\sin\theta$. This gives

$$u_{\pi/2}(t)\cos t = u_{\pi/2}(t)\sin\left(\frac{\pi}{2} - t\right) = -u_{\pi/2}(t)\sin\left(t - \frac{\pi}{2}\right)$$

Substituting this into the ODE and taking the Laplace transform gives us

$$(s^2+4)Y(s) = \frac{1}{s^2+1} - \frac{e^{-\frac{\pi}{2}s}}{s^2+1}$$

Solving for Y yields

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)}(1 - e^{-\frac{\pi}{2}s})$$

Let

$$H(s) = \frac{1}{(s^2+1)(s^2+4)}$$

We use partial fractions to rewrite H:

$$\begin{aligned} \frac{1}{(s^2+1)(s^2+4)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \\ \Longrightarrow 1 &= (As+B)(s^2+1) + (Cs+D)(s^2+4) \\ \Longrightarrow 1 &= (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D) \end{aligned}$$

Equating coefficients yields the four equations

$$A + C = 0, \;\; B + D = 0, \;\; A + 4C = 0, \;\; B + 4D = 1$$

which has the solution $A = C = 0$, $B = -1/3$ and $D = 1/3$. Therefore we can write

$$H(s) = -\frac{1}{6} \cdot \frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1}$$

and so

$$h(t) = \mathcal{L}^{-1}\{H\} = -\frac{1}{6}\sin(2t) + \frac{1}{3}\sin(t).$$

It follows (using the t-shifting theorem where necessary) that

$$y(t) = h(t) - u_{\pi/2}(t)h(t - \pi/2)$$

Problem 5: Determine the value of the following integrals

(a) $\int_2^7 \delta(t+1)\, dt$.

(b) $\int_{-2}^7 \delta(t-1)\, dt$.

(c) $\int_1^8 \ln(t^2)\delta(t-e)\, dt$.

(d) $\int_{-2}^4 (2t^4 - t^3 + 7t^2 - 1)\, \delta(t-5)\, dt$.

Solution:

To evaluate the above integrals, we will be using the well-known formula derived in class which is given by

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0).$$

It should be noted in passing that the value of t_0 must lie within the interval of integration. Otherwise, the value of the definite integral will be zero.

(a) 0

(b) 1

(c) 2

(d) 0

Problem 6: Find the solution of the initial value problem and *sketch the graph* of the solution for

$$y'' + 2y' + 2y = \delta(t-\pi); \quad y(0) = 1, y'(0) = 0. \tag{0.1}$$

Solution:

Let us take initially the Laplace transform of the IVP given and solve for $Y(s) = \mathcal{L}\{y(t)\}$ afterwards:

$$s^2Y - s + 2sY - 2 + 2Y = e^{-\pi s} \Rightarrow Y = \frac{s+2}{s^2+2s+2} + \frac{e^{-\pi s}}{s^2+2s+2} \Rightarrow$$
$$Y = \frac{s+2}{(s+1)^2+1} + \frac{e^{-\pi s}}{(s+1)^2+1} \Rightarrow$$
$$Y = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + \frac{e^{-\pi s}}{(s+1)^2+1}. \tag{0.2}$$

Next, note that the inverse of the first two terms in Eq. (0.2) can be obtained by utilizing the s-shifting theorem, that is,

$$\mathcal{L}\left\{\frac{s+1}{(s+1)^2+1}\right\}^{-1} = e^{-t}\,\mathcal{L}\left\{\frac{s}{s^2+1}\right\}^{-1} = e^{-t}\cos(t), \tag{0.3}$$
$$L\left\{\frac{1}{(s+1)^2+1}\right\}^{-1} = e^{-t}\,\mathcal{L}\left\{\frac{1}{s^2+1}\right\}^{-1} = e^{-t}\sin(t). \tag{0.4}$$

Note that we could use the formulas 9 and 10 in p.g. 252 in the textbook for the above inverses.

Finally, and as far as the last term in Eq. (0.2) is concerned, we will apply a combination of the t- and s- shifting theorems:

$$\mathcal{L}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}^{-1} = \mathcal{L}\left\{H(s+1)e^{-\pi s}\right\}^{-1} = u_\pi(t)h(t-\pi), \tag{0.5}$$

where $h(t) = e^{-t}\sin(t)$ and u_π represents the Heaviside function for $c = \pi$. As a side note, $H(s) = \frac{1}{s^2+1}$ and $h(t) = \mathcal{L}\{H(s+1)\}^{-1} = e^{-t}\sin(t)$, see Eq. (0.4).

Thus, upon combining all the above terms, the solution to the IVP is given by

$$\boxed{y = y(t) = e^{-t}\left[\cos(t) + \sin(t)\right] - u_\pi(t)e^{-(t-\pi)}\sin(t)}. \tag{0.6}$$

The graph of the solution given by Eq. (0.6) is shown in Fig. 4.

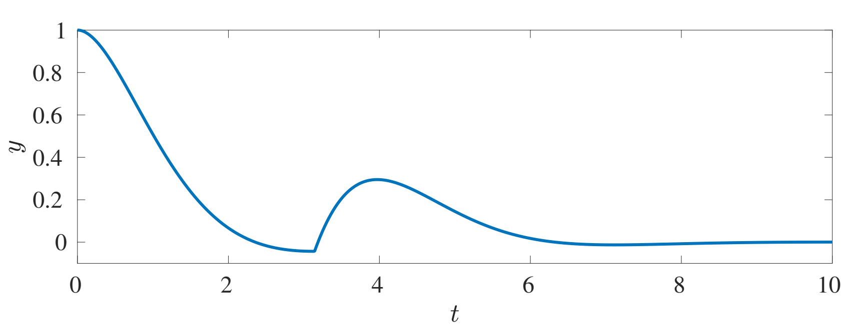

FIGURE 4. Graph of the solution to the IVP of Question 1.

Problem 7: Find the solution of the initial value problem and sketch the graph of the solution for

$$y'' + y = \delta(t - 2\pi)cos(t); \quad y(0) = 0, y'(0) = 1$$

Solution:

Take the Laplace transform of the equation to get

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \mathcal{L}\{\delta(t - 2\pi)cos(t)\}$$

To find $\mathcal{L}\{\delta(t - 2\pi)cos(t)\}$, by definition of the Laplace transform and use the property of delta function to get

$$\mathcal{L}\{\delta(t - 2\pi)cos(t)\} = \int_0^\infty \delta(t - 2\pi)cos(t)e^{-st}dt = cos(2\pi)e^{-2\pi s} = e^{-2\pi s}$$

Plugging initial values yields

$$(s^2 + 1)Y(s) - 1 = e^{-2\pi s}$$

Solving for Y(s) to get

$$Y(s) = \frac{e^{-2\pi s}}{s^2 + 1} + \frac{1}{s^2 + 1}$$

Let $H(s) = \frac{1}{s^2+1}$, then $h(t) = \mathcal{L}^{-1}\{H\} = sin(t)$. Apply t-shifting theorem to get

$$\mathcal{L}^{-1}\{e^{-2\pi s}H(s)\} = u_{2\pi}(t)h(t - 2\pi) = u_{2\pi}(t)sin(t - 2\pi) = u_{2\pi}(t)sin(t)$$

So the solution is

$$y(t) = \mathcal{L}^{-1}\{Y\} = u_{2\pi}(t)sin(t) + sin(t)$$

The graph is sketched in figure 5

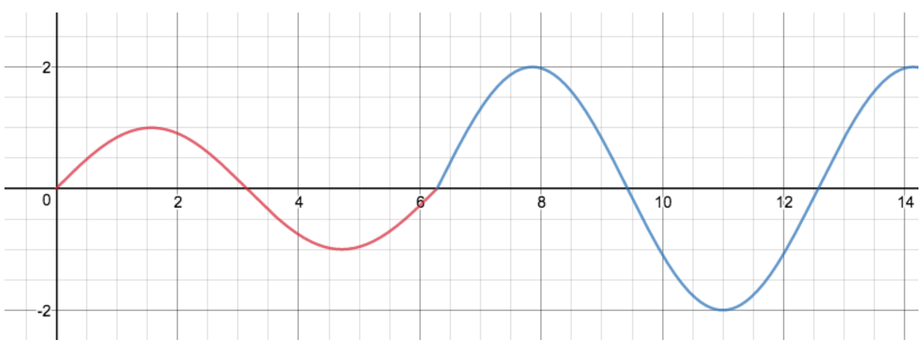

FIGURE 5. Graph of the solution

Problem 8: Consider the initial value problem

$$y'' + 2y' + y = k\delta(t - 1), \ \ y(0) = 0, \ y'(0) = 0.$$

(a) Find the solution of the initial value problem.

(b) Find the value of k for which the response has a peak value of 2.

Solution:

(a) Take the Laplace transform of the differential equation to get

$$(s^2 + 2s + 1)Y(s) = ke^{-s}$$

so that after noting that $s^2 + 2s + 1 = (s + 1)^2$ and solving for Y we have

$$Y(s) = ke^{-2}\frac{1}{(s+1)^2}.$$

We recognize $1/(s+1)^2$ as $1/s^2$ shifted by the value $c = -1$. Applying the s-shifting theorem gives us

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

Now applying the t-shifting theorem gives us the solution

$$y(t) = k \cdot u_1(t) \cdot (t-1)e^{-t+1}$$

(b) As $y(t) = 0$ for $t < 1$, we ignore this region and look for extreme values on the interval $t \geq 1$. For this time range, we may replace $u_1(t)$ with 1. Now

$$y'(t) = ke^{-t+1} - k(t-1)e^{-t+1}$$

and setting equal to zero gives the equation

$$ke^{-t+1} - k(t-1)e^{-t+1} = 0 \implies ke^{-t+1}(2-t) = 0$$

so that we conclude the extreme value occurs at $t = 2$. Computing $y(2)$ and setting equal to 2 gives us the equation for k:

$$\frac{k}{e} = 2 \implies k = 2e$$