1. A curve is given by the equation $x^2 + xy + y^2 = 3$.

(a) (10) Compute the derivative $\frac{dy}{dx}$ of the curve at the point $(1, 1)$.

**ANS:**

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$

or

$$2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x + y}{x + 2y}$$ (6 pts)

and

$$\frac{dy}{dx} \bigg|_{(1,1)} = -\frac{2x + y}{x + 2y} \bigg|_{(1,1)} = -\frac{1}{1} = -1$$ (4 pts)

(b) (10) Find the points where the tangent to the curve is horizontal.

**ANS:**

The tangent line is horizontal at a point $(x, y)$ when

$$\frac{dy}{dx} = 0 \Rightarrow -\frac{2x + y}{x + 2y} = 0 \Rightarrow 2x + y = 0 \Rightarrow y = -2x$$ (4 pts)

Now, $(x, y)$ must also satisfy the equation for the curve, so

$$x^2 + x(-2x) + (-2x)^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$ (4 pts)

This gives the $x$ values, so the points are $(1, -2)$ and $(-1, 2)$ (2 pts).

2. Differentiate the following functions

(a) (10) $f(x) = \sqrt{\ln(\tan(x))}$

**ANS:**

$$\frac{d}{dx} \sqrt{\ln(\tan(x))} = \frac{d}{dx} (\ln(\tan(x)))^{1/2}$$

$$= \frac{1}{2} (\ln(\tan(x)))^{-1/2} \frac{1}{\tan(x)} \sec^2(x).$$
(b) (10) \( f(x) = x^{6x}e^{x^2-1} \)

**ANS:** There is no way to do this problem without logarithmic differentiation.

\[
f(x) = x^{6x}e^{x^2-1}
\]

\[
\ln(f(x)) = \ln(x^{6x}e^{x^2-1}) = \ln(x^{6x}) + \ln(e^{x^2-1}) = 6x \ln(x) + x^2 - 1
\]

\[
\frac{d}{dx} \ln(f(x)) = \frac{d}{dx}(6x \ln(x)) + \frac{d}{dx}(x^2 - 1)
\]

\[
= \frac{d}{dx}(6x \ln(x)) + 2x \quad \text{Use the product rule on the first term.}
\]

\[
= 6x \frac{1}{x} + 6 \ln(x) + 2x
\]

\[
\frac{f'(x)}{f(x)} = 6 + 6 \ln(x) + 2x \quad \text{Multiplying both sides by } f(x)
\]

\[
f'(x) = f(x)(6 + 6 \ln(x) + 2x)
\]

\[
f'(x) = x^{6x}e^{x^2-1}(6 + 6 \ln(x) + 2x)
\]

You may also use the product rule with \( u = x^{6x} \) and \( v = e^{x^2-1} \). But you will still need logarithmic differentiation to find \( u' = x^{6x} \left(6x \frac{1}{x} + 6 \ln(x)\right) \) see example 8 in section 3.8 (pg. 247) in the text book for a calculation similar to the one for \( u' \). You will also need to find \( v' = 2xe^{x^2-1} \) using the chain rule.

3. Let \( f(x) = e^{3x} + \sin(x) \).

(a) (12) Compute the first three derivatives \( f'(x), f''(x), f'''(x) \).

**ANS:**

\[
f'(x) = 3e^{3x} + \cos(x),
\]

\[
f''(x) = 3(3e^{3x}) - \sin(x) = 9e^{3x} - \sin(x),
\]

\[
f'''(x) = 9(3e^{3x}) - \cos(x) = 27e^{3x} - \cos(x).
\]

(b) (8) Find \( f^{(37)}(0) \).
ANS:

\[ f^{(37)}(x) = 3^{37} e^{3x} + \cos(x), \]
\[ f^{(37)}(0) = 3^{37} e^0 + \cos(0) = 3^{37} + 1. \]

4. In a building which is 100 ft high, a woman takes an elevator at the top of the building and moves downward at a speed of 16 ft/sec. At exactly the same time a man exits the building and travels along a straight line at a speed of 3ft/sec. Find the rate of increase of the distance between the man and woman after 5 seconds.

ANS:
Let \( x \) be the (horizontal) distance between the bottom of the building and the man and let \( y \) be the (vertical) distance between the bottom of the building and the woman.

The distance \( z \) between the man and the woman is related to \( x \) and \( y \) by

\[ z^2 = x^2 + y^2. \]

We know that at time 0 we have \( x(0) = 0 \) and \( y(0) = 100 \) and that

\[ \frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dy}{dt} = -16 \]

(Be careful about the sign!)

After 5 seconds, we have \( x(5) = 3 \times 5 = 15 \) and \( y(5) = 100 - 5 \times 16 = 20 \).

Therefore the distance between the man and the woman after 5 seconds is

\[ \sqrt{20^2 + 15^2} = \sqrt{625} = 25. \]

If we differentiate the relation \( z^2 = x^2 + y^2 \) with respect to \( t \) we find

\[ 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}, \]

or

\[ \frac{dz}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z}. \]

After 5 seconds

\[ \frac{dz}{dt} = \frac{15 \times 3 + 20 \times (-16)}{25} = -11. \]

5. If a piece of chalk is thrown vertically upward with a velocity of 32ft/sec, then the height after the \( t \) seconds is

\[ s(t) = 32t - 16t^2. \]
(a) (4) Find the velocity of the piece of chalk after 2 seconds.

**ANS:**

\[ v(t) = 32 - 32t \]

so that \( v(2) = 32 - 64 = -32 \). The velocity after 2 seconds is \(-32\text{ft/sec}\).

(b) (4) When is the piece of chalk at rest?

**ANS:**

At rest when \( v(t) = 32(1 - t) = 0 \), i.e., \( t = 1 \).

(c) (4) What is the acceleration?

**ANS:**

\[ a(t) = -32 \]

(d) (4) When is the piece of chalk speeding up/slowing down?

**ANS:**

Speeding up if \( a < 0, v < 0 \), i.e., for \( 1 < t \). Slowing down if \( a < 0, v > 0 \), i.e., for \( 1 > t \).

(e) (4) What is the velocity of the piece of chalk when it is 12 ft above the ground on its way up?

**ANS:**

\[ s(t) = 32t - 16t^2 = 12 \]

if \( t^2 - 2t + \frac{3}{4} = 0 \) which gives

\[ t_1 = \frac{1}{2}, t_2 = \frac{3}{2} \].

In the first case \( v\left(\frac{1}{2}\right) = 16 \), in the second \( v\left(\frac{3}{2}\right) = -16 \). This shows that the chalk is on its way up when \( t = \frac{1}{2} \) and then the velocity is 16ft/sec.