1. a: Solve the equation using row operations

\[
\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} y + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} z = \begin{pmatrix} -2 \\ 7 \\ 1 \end{pmatrix}.
\]

Set up the augmented matrix and use Gaussian elimination:

\[
\begin{pmatrix} 1 & 0 & -1 & : & -2 \\ 0 & 2 & 1 & : & 7 \\ -1 & 1 & 0 & : & 1 \end{pmatrix} \rightarrow \text{Show your work} \rightarrow
\begin{pmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{pmatrix}.
\]

Thus the solution is \[\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.
\]

b: Compute the dot product of the vectors \( u = \begin{pmatrix} 1 \\ 0 \\ -3 \\ -1 \end{pmatrix}, v = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix} \) Are these vectors perpendicular? Why?

\[
u \cdot v = 1(-1) + 0(-2) + (-3)3 + (-1)(1) = -11.
\]

Since the dot product is not 0, the vectors are not perpendicular.

c: Compute the matrix products \( AB \) and \( BA \) if possible:

\[
A = \begin{pmatrix} -1 & 0 & 3 \\ -2 & 5 & 3 \\ 0 & 3 & 1 \end{pmatrix}
\]

and

\[
B = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 4 \end{pmatrix}.
\]

The product \( AB \) does not make sense because the number of columns of \( A \) does not equal the number of rows of \( B \). The product \( BA \) is

\[
BA = \begin{pmatrix} -1 & 21 & 10 \\ -4 & 27 & 7 \end{pmatrix}.
\]
d: For what vectors \( v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) does the equation \( Ax = v \) have a solution if
\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & -1 & 2 \\
1 & -1 & 3
\end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
Set up the augmented matrix and use Gaussian elimination:
\[
\begin{pmatrix}
1 & 0 & 1 & : & a \\
0 & -1 & 2 & : & b \\
1 & -1 & 3 & : & c
\end{pmatrix} \rightarrow \text{Show your work} \rightarrow
\begin{pmatrix}
1 & 0 & 1 & : & a \\
0 & 1 & -2 & : & -b \\
0 & 0 & 0 & : & c - a - b
\end{pmatrix}.
\]
It follows that the system has solutions if \( c - a - b = 0 \).

2. Let \( A \) be the matrix
\[
\begin{pmatrix}
4 & 2 & -4 \\
2 & 1 & -2 \\
-4 & -2 & 4
\end{pmatrix}.
\]
This represents a linear transformation \( T \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) with respect to the basis
\[
E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]
Let \( F \) be the basis of \( \mathbb{R}^3 \) given by \( \left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \). The inverse of the matrix
\[
\begin{pmatrix}
2 & 0 & 1 \\
1 & 2 & 0 \\
-2 & 1 & 1
\end{pmatrix}
\]
is
\[
\begin{pmatrix}
2/9 & 1/9 & -2/9 \\
-1/9 & 4/9 & 1/9 \\
5/9 & -2/9 & 4/9
\end{pmatrix}.
\]
a: Let \( w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). What are the coordinates of \( w \) with respect to \( F \).
\[
[w]_F = \begin{pmatrix}
2/9 & 1/9 & -2/9 \\
-1/9 & 4/9 & 1/9 \\
5/9 & -2/9 & 4/9
\end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 4/9 \\ 7/9 \end{pmatrix}.
\]
b: What is the matrix of the linear transformation \( T \) with respect to the basis \( F \). Write your answer as a product of matrices, but do not compute the product.
3. a: What does it mean for a basis of $\mathbb{R}^3$ to be orthonormal.

A basis \{\(u_1, u_2, u_3\) of $\mathbb{R}^3$ is orthonormal if $u_i \cdot u_j = 0$ for $1 \leq i \neq j \leq 3$ and $u_i \cdot u_i = 1$ for $1 \leq i \leq 3$. b: Let $F = \{f_1, f_2, f_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let $T$ be the reflection across the plane spanned by $f_1$ and $f_2$. What is the matrix of $T$ with respect to the basis $F$?

$$[^T]_{F-F} = [I]_{F-E}[^T]_{E-E}[I]_{E-F}$$

$$= \begin{pmatrix} 2/9 & 1/9 & -2/9 \\ -1/9 & 4/9 & 1/9 \\ 5/9 & -2/9 & 4/9 \end{pmatrix} \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

4. Let $P_2$ be the vector space of polynomials of degree less than or equal to 2. Which of the following are subspaces of $P_2$. Explain your reasoning. These explanations are the most important part of the questions.
a: \(\{f \in P_2 | f(-1) = 1\}\)

This is not a subspace of $P_2$ because it does not contain the zero polynomial $Z(x) = 0$.
b: \(\{f \in P_2 | f(-1) = 0\}\)

This is a subspace of $P_2$. It clearly contains $Z(x) = 0$. If $f$ and $g$ are in this set then

$$(f + g)(-1) = f(-1) + g(-1) = 0 + 0 = 0,$$

hence $f + g$ is in the set. Finally, if $f$ is in the set and $c$ is a scalar,

$$(cf)(-1) = cf(-1) = c \cdot 0 = 0,$$

hence $cf$ is in the set.
c: \(\{f \in P_2 | f'' + 2f' = 0\}\)

This is a subspace of $P_2$. It clearly contains $Z(x) = 0$. If $f$ and $g$ are in this set then

$$(f + g)'' + 2(f + g)' = f'' + g'' + 2f' + 2g' = 0 + 0 = 0,$$
hence \( f + g \) is in the set. Finally, if \( f \) is in the set and \( c \) is a scalar,
\[
(cf)'' + 2(cf)' = cf'' + 2cf' = c(f'' + 2f') = c \cdot 0 = 0,
\]
hence \( cf \) is in the set.

5. a: Define what it means for a subset of a vector space to be a basis of that subspace.
A subset \( B \) of a subspace \( V \) is a basis for \( V \) if \( B \) is linearly independent and \( \text{span}(B) = V \).

b: Is \( \{1, (t - 1), (t - 1)^2, (t - 1)^3\} \) a basis of \( P_3 \)? Here \( P_3 \) denotes the vector space of all polynomials of degree less than or equal to 3. Why? Again note that explaining why is the important part of the question.
Since \( P_3 \) is 4 dimensional, it suffices to show that \( B = \{1, (t - 1), (t - 1)^2, (t - 1)^3\} \) is linearly independent. Suppose that \( \phi(t) = a1 + b(t - 1) + c(t - 1)^2 + d(t - 1)^3 \). We need to show that if \( \phi \) is the zero function then \( a = b = c = d = 0 \). Suppose \( \phi(t) = 0 \).

Thus \( a = b = c = d = 0 \).

6. a: Let \( V, W \) be vector spaces. Define what it means for a function \( F : V \to W \) to be a linear transformation.
\( F \) is a linear transformation if

(a) \( F(0_V) = 0_W \), where \( 0_V \) is the zero vector in \( V \) and \( 0_W \) is the zero vector in \( W \).

(b) \( F(f + g) = F(f) + F(g) \) for every \( f, g \in V \).

(c) \( F(cf) = cF(f) \) for every \( f \in V \) and \( c \in \mathbb{R} \).

b: Are the following linear transformations? Why? Note that the why part of the question is very important.
b1: \( F : P_2 \to P_2, p \mapsto p'' - 3p \)
\( F \) is a linear transformation. To see this, we check the definition:

(a) \( F(Z) = Z'' - 3Z = Z \), where \( Z \) is the zero polynomial \( Z(t) = 0 \).

(b) For every \( f, g \in P_2 \),
\[
F(f + g) = (f + g)'' - 3(f + g) \\
= f'' + g'' - 3f - 3g \\
= f'' - 3f + g'' - 3g \\
= F(f) + F(g).
\]
(c) For every \( f \in P_2, c \in \mathbb{R} \),
\[
F(cf) = (cf)'' - 3(cf) \\
= cf'' + 3cf \\
= c(f'' - 3f) \\
= cF(f).
\]

b2: \( F : P_2 \to \mathbb{R}, p \mapsto p(2) \)

\( F \) is a linear transformation. To see this, we check the definition:

(a) \( F(Z) = Z(2) = 0 \), where \( Z \) is the zero polynomial \( Z(t) = 0 \).
(b) For every \( f, g \in P_2 \),
\[
F(f + g) = (f + g)(2) \\
= f(2) + g(2) \\
= F(f) + F(g).
\]
(c) For every \( f \in P_2, c \in \mathbb{R} \),
\[
F(cf) = (cf)(2) \\
= cf(2) \\
= cF(f).
\]

b3: \( F : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}, A \mapsto A^2 - A \). \( F \) is not a linear transformation. We can easily check that \( F \) does not respect the scalar multiplication. In particular, let \( I \) be the \( 2 \times 2 \) identity matrix. Then one computes that \( F(2I) = 3I \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 2F(I) \).

7. Consider the differential equation \( y'' + y = 0 \).

a: The functions \( y_1(t) = \cos(t), y_2(t) = \sin(t) \) are solutions to this differential equation. Let \( V \) be the vector space of functions from \( \mathbb{R} \) to \( \mathbb{R} \) that have arbitrarily many derivatives. Let \( D : V \to V \) be defined by \( D : f \mapsto f'' + f \). What properties of \( D \) insure that the functions \( a_1y_1 + a_2y_2 \) for any \( a_1, a_2 \in \mathbb{R} \) also solutions to our differential equation?

The fact that \( D \) is a linear transformation from \( C^\infty \to C^\infty \) guarantees that \( a_1y_1 + a_2y_2 \) is a solution for any \( a_1, a_2 \in \mathbb{R} \).

\[
D(a_1y_1 + a_2y_2) = a_1D(y_1) + a_2D(y_2) \quad \text{by the linearity of } D. \\
0 + 0, \quad \text{since } y_1, y_2 \text{ are solutions to the differential equation.} \\
= 0
\]

Another way to phrase this is as follows. Since \( D \) is a linear transformation, the kernel of \( D \) is a subspace. In particular, \( \ker(D) \) is closed under taking linear combinations. We note that \( \ker(D) \) is precisely the space of solutions to our differential equation. Thus \( y_1, y_2 \in \ker(D) \) implies \( a_1y_1 + a_2y_2 \in \ker(D) \).
b: The function \( y(t) = t + 1 \) is a solution to the differential equation \( y'' + y = t + 1 \) (We are giving you this fact; you do not have to show this is the case). What are all the solutions to the differential equation \( y'' + y = t + 1 \). The solutions are
\[
\{ t + 1 + a_1y_1 + a_2y_2 \mid a_1, a_2 \in \mathbb{R} \}.
\]
c: Justify your answer to part (b) of this question.
We use the linearity of \( D \). Suppose \( f \) is a solution to \( y'' + y = t + 1 \). Then \( f(t) = t + 1 + \text{junk} \). For \( f \) to be a solutions, we see that
\[
t + 1 = D(f) = D(t + 1 + \text{junk}) = D(t + 1) + D(\text{junk}) = t + 1 + D(\text{junk}).
\]
Thus for \( f \) to be a solution, we need that \( \text{junk} \in \ker(D) \), which was computed in part a: to be \( a_1y_1 + a_2y_2 \ | \ a_1, a_2 \in \mathbb{R} \).

8. a: State the rank-nullity theorem. Define rank and nullity.
Let \( T : V \to W \) be a linear transformation between vector spaces \( V \) and \( W \). Then
\[
\text{rank}(T) + \text{null}(T) = \dim(V).
\]
The rank of \( T \), denoted \( \text{rank}(T) \) is the dimension of the image of \( T \). The nullity of \( T \), denoted \( \text{null}(T) \) is the dimension of the kernel of \( T \).
b: Show that any differential equation of the form
\[
\frac{d^2y}{dx^2} + y = f(x)
\]
has a solution for any polynomial \( f(x) \in P_2 \).
Let \( D \) be the linear transformation that sends \( g \mapsto g'' + g \). We are being asked to show that \( D(y) = f \) has a solution for every \( f \in P_2 \). In other words, we need to show that every \( f \in P_2 \) is in the image of \( D \). View \( D \) as a linear transformation from \( P_2 \) to \( P_2 \). Let \( B = \{1, x, x^2\} \) be a basis for \( P_2 \). Then in these coordinates,
\[
[D]_{B \to B} = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
which clearly has rank 3. In particular, every \( f \in P_2 \) is in the image of \( D \) and hence for every \( f \in P_2 \), \( D(y) = f \) has a solution.
9. Let $T$ be the linear transformation from $P_2$ to $P_2$ given by

$$f(x) \mapsto f'' - 2f.$$ 

Find the matrix of $T$ with respect to the basis $\mathcal{B} = \{1, x - 1, (x - 1)^2\}$.

$$[T]_{\mathcal{B} \rightarrow \mathcal{B}} = \begin{pmatrix}
\vdots & \vdots & \vdots \\
[T(1)]_{\mathcal{B}} & [T(x - 1)]_{\mathcal{B}} & [T((x - 1)^2)]_{\mathcal{B}} \\
\vdots & \vdots & \vdots \\
[-2]_{\mathcal{B}} & [-2(x - 1)]_{\mathcal{B}} & [2 - 2(x - 1)^2)]_{\mathcal{B}} \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix}.$$ 

10. Define an inner product (or equivalently, a dot product) on $P_2$ by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$.

Find an orthonormal basis of $P_2$ with respect to this inner product.

We first find a basis of $P_2$, then use Gram-Schmidt to create an orthonormal basis. Fix a basis $\mathcal{B} = \{f_1(x) = 1, f_2(x) = x - \frac{1}{2}, f_3(x) = (x - \frac{1}{2})^2\}$. Recall that the Gram-Schmidt process yields an orthonormal basis $\{u_1, u_2, u_3\}$, where

$$u_1 = \frac{1}{\|f_1\|} f_1,
\quad u_2 = \frac{1}{\|f_2\|} f_2^\perp, \quad \text{where } f_2^\perp = f_2 - \langle f_2, u_1 \rangle u_1
\quad u_3 = \frac{1}{\|f_3\|} f_3^\perp, \quad \text{where } f_3^\perp = f_3 - \langle f_3, u_1 \rangle u_1 - \langle f_3, u_2 \rangle u_2.$$ 

We use the definitions to compute the relevant inner products:

$$\|f_1\|^2 = \langle f_1, f_1 \rangle = \int_0^1 1\,dx = 1,$$

and so $u_1(x) = 1$. To compute $u_2$,

$$\langle f_2, u_1 \rangle = \int_0^1 x - \frac{1}{2}\,dx = 0, \quad \text{hence}$$

$$f_2^\perp(x) = x - \frac{1}{2} \quad \text{and}$$

$$\|f_2^\perp\|^2 = \langle f_2^\perp, f_2^\perp \rangle = \int_0^1 (x - \frac{1}{2})^2\,dx = \frac{1}{12}.$$
It follows that
\[ u_2(x) = 2\sqrt{3}(x - \frac{1}{2}). \]

to compute \( u_3 \),
\[ \langle f_3, u_1 \rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12} \quad \text{and} \]
\[ \langle f_3, u_2 \rangle = 2\sqrt{3} \int_0^1 (x - \frac{1}{2})^3 dx = 0, \quad \text{hence} \]
\[ f_3^\perp(x) = (x - \frac{1}{2})^2 - \frac{1}{12} \quad \text{and} \]
\[ \|f_3^\perp\|^2 = \langle f_3^\perp, f_3^\perp \rangle = \int_0^1 ((x - \frac{1}{2})^2 - \frac{1}{12})^2 dx = \frac{7}{20}. \]

It follows that
\[ u_3(x) = \frac{2\sqrt{5}}{\sqrt{7}} ((x - \frac{1}{2})^2 - \frac{1}{12}). \]

11. The two vectors \( u_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \) and \( u_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \) are orthogonal.

a: Verify this fact.

To verify this fact, we compute the dot product to see that it is 0.
\[ u_1 \cdot u_2 = 1(1) + (-1)(3) + (2)(1) = 0. \]

b: Find the vector in the space spanned by \( u_1 \) and \( u_2 \) that is closest to the vector \( \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \).

Let \( v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) and \( V \subset \mathbb{R}^3 \) the space spanned by \( u_1 \) and \( u_2 \). Then vector in \( V \) that is closest to \( v \) is the projection of \( v \) onto the \( V \). Since \( u_1 \) and \( u_2 \) are orthogonal, we can get an orthonormal basis of \( V \) by dividing by the respective lengths. In particular,
\[ \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\} \]
is an orthonormal basis of $V$.

We compute the projection of $v$ onto $V$ is then

$$(v \cdot \tilde{u}_1)\tilde{u}_1 + (v \cdot \tilde{u}_2)\tilde{u}_2 = \begin{pmatrix} \frac{1}{6} - \frac{8}{11} \\ \frac{1}{3} + \frac{8}{11} \end{pmatrix} = \begin{pmatrix} \frac{37}{66} \\ \frac{243}{66} \end{pmatrix}.$$

12. Using the method of expansion by minors compute the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Note that you are told to use expansion by minors, and so you must use that method to receive credit.

$$\det \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 1 \det \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} - 0 \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$$

$$= -4 + 0 + 3(2)$$

$$= 2.$$

13. a: Define eigenvalue and eigenvector of a matrix.

Let $A$ be a $n \times n$ matrix. Then a non-zero vector $v \in \mathbb{R}^n$ is an eigenvector of $A$ with associated eigenvalue $\lambda$ if

$$Av = \lambda v.$$

b: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 7/5 & -1/5 \\ -6/5 & 8/5 \end{pmatrix}.$$ 

The eigenvalues are the roots of the characteristic polynomial

$$f_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 7/5 - \lambda & -1/5 \\ -6/5 & 8/5 - \lambda \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

It follows that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. An eigenvector $v_1$ associated to $\lambda_1 = 1$ is a non-zero vector in $\ker(A - I) = \ker \begin{pmatrix} 2/5 & -1/5 \\ -6/5 & 3/5 \end{pmatrix}$. By inspection,

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector associated to eigenvalue 1. Similarly, an eigenvector $v_2$
associated to $\lambda_2 = 2$ is a non-zero vector in $\ker(A - 2I) = \ker \begin{pmatrix} -3/5 & -1/5 \\ -6/5 & -2/5 \end{pmatrix}$. By inspection, $v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is an eigenvector associated to eigenvalue 2.

c: Find a matrix $B$ so that $C = BAB^{-1}$ is diagonal. What is the matrix $C$?

Partial Answer: the eigenvalues of the matrix $A$ are 1, 2, corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Note that $A$ is diagonalizable, and an eigenbasis, $B = \{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \}$ are the coordinates in which $A$ looks diagonal. In particular,

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = S^{-1}AS, \quad \text{where } S = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$ 

It follows that the matrix $C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = S^{-1} = \begin{pmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{pmatrix}$.

14. Let $C(n)$ denote the coyote population of an imaginary ecosystem after $n$ time intervals have passed. Similarly let $R(n)$ denote the population of this ecosystem after $n$ time intervals. Let

$$A = \begin{pmatrix} 10/21 & 16/21 \\ -8/21 & 46/21 \end{pmatrix}.$$ 

Assume that

$$\begin{pmatrix} C(n) \\ R(n) \end{pmatrix} = A \begin{pmatrix} C(n + 1) \\ R(n + 1) \end{pmatrix}.$$ 

Describe how this ecosystem behaves for different positive initial values $\begin{pmatrix} R(0) \\ C(0) \end{pmatrix}$. In particular indicate for what initial values both species survive and for what initial values only one species survives in the long run.

Partial answer: The eigenvalues of the matrix $A$ are $2/3, 2$. The corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Suppose the initial population is $\begin{pmatrix} C_0 \\ R_0 \end{pmatrix}$. We express this in terms of the eigenvectors as

$$\begin{pmatrix} C_0 \\ R_0 \end{pmatrix} = a \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$ 

Then

$$\begin{pmatrix} C(t) \\ R(t) \end{pmatrix} = A^t \begin{pmatrix} C_0 \\ R_0 \end{pmatrix} = a \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} t^t \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b(2^t) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
As $t$ gets large, $(\frac{2}{3})^t$ gets small and $2^t$ gets large, so we see that in order for the populations to both survive, the initial population $\begin{pmatrix} C_0 \\ R_0 \end{pmatrix} = a \begin{pmatrix} 4 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4a + 2b \\ a + b \end{pmatrix}$ must be such that $b > 0$. This is precisely when $C_0 < 4R_0$. 

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