Homework Problem Set 5

1. One of the main issues that arose in the midterm exam is that it is not completely clear how to develop an implicit solver of our PDEs. Hence, it is relevant to try to build a simple implicit integrator.

   - As our simplest example, let’s take the Euler in time, implicit centered difference in space scheme for the solution of the 1d diffusion PDE \( u_t = u_{xx} \). Find (with paper and pencil) the actual solution of the numerical scheme for
     \[ u(x, 0) = \sin\left(\frac{\pi x}{4}\right) + 4\sin\left(\frac{\pi x}{2}\right). \]
     in the domain \([0, 4]\) for Dirichlet boundary conditions.

   - Now write a numerical code that finds this solution and make sure that the two are identical. Also show the solution as output in a space time 3d plot using mesh and in a contour plot using imagesc.


   - Prove his statement for this spherically symmetric case that the scheme is second order accurate in both space and time (the statement is right under Eq. (3.2c)).

   - Try to simulate the collapse of the so-called \( \phi^4 \) domain wall of Eq. (3.3) [the tanh solution] towards a black hole for the potential of Eq. (2.1) [again the lower, \( \phi^4 \) potential of the equation]. Use \( dr = 0.025 \) and \( dt = 0.001 \), \( R_0 = 10 \), \( m = 1 \) and \( \lambda = 6.25 \) for your simulations which should give results similar (but probably not identical since the author does not specify precisely all parameters) to the left panel of Figure 4. Use imagesc to show the time evolution of your simulation. Your initial condition for \( \phi \) should be given by the (lower) domain wall of (3.3) and your initial condition for \( \dot{\phi} \) should be 0. Choose boundary conditions as you find appropriate.