Homework Problem Set 1

1. Differential and Difference Equations:

   - Solve the difference equation $a_{n+3} - a_n = n + 4$.
   - Show that if $\lambda_0$ is a double root of the characteristic polynomial $\sum_{n=0}^{N} a_n \lambda^n = 0$, then the corresponding ODE $\sum_{n=0}^{N} a_n y^{(n)} = 0$, has both the root $e^{\lambda_0 x}$ and the root $xe^{\lambda_0 x}$.

2. Numerical Solution of ODEs:

   - Consider the ODE $y' = x^3 - y$ with $y(0) = 1$. Find its exact solution.
   - Find its approximate solution numerically using explicit Euler and stepsizes 0.1 and 0.01. Show a graph of all 3 solutions (up to $t = 5$) and a graph of the difference between each of the two numerical solutions and the analytical solution.
   - Consider the second order ODE $y'' + 2y' + y = 2\cos(x)$, with $y(0) = 0$ and $y'(0) = 1$. Find its solution.
   - Implement a second order Runge-Kutta scheme to solve the equation and compare your numerical solution with the analytical solution as in the previous problem.

3. Schemes and Truncation Errors:

   - Consider the two-step method
     \[ y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4} (4y'_{n+1} - y'_{n} + 3y'_{n-1}) . \]
     Find its order and its truncation error.
   - Use the quadratic interpolant to $Y' = f(x, Y)$ at $x_n, x_{n-1}, x_{n-2}$ to get the formula:
     \[ Y_{n+1} = Y_{n-3} + \frac{h}{3} (2Y'_{n} - Y'_{n-1} + 2Y'_{n-2}) + \frac{14}{45} h^5 Y^{(5)}(\xi). \]