Homework Problem Set 4

1. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
   - Find its eigenvalues and eigenvectors.
   - Use the methods we discussed in class to show that as $\nu \to \infty$ for a power method iteration, the normalization factor $\sigma_\nu \to |\lambda_1|$, where $\lambda_1$ is the dominant eigenvalue.
   - Combining the power method and the above observation, write a short function that performs a power method for a matrix $A$ with an initial guess $X$ for the dominant eigenvector, up to a tolerance $tol$ and with a maximal number of iterations $\text{max.it}$.
   - Apply the method to the above matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, with an initial guess of $X = [1, 0]'$ and show the results (say for $tol = 1e-08$ and $\text{max.it} = 1000$).

2. Consider a matrix $A$ with eigenvalues $\lambda_i$ and eigenvectors $X^i$. Consider for simplicity that the eigenvectors are orthonormal so that the inner product $(X^i, X^j) = (X^i)^T X^j = \sum_{k=1}^n X^i_k X^j_k = \delta_{ij}$ If the dominant eigenvalue is $\lambda_1$ and the corresponding eigenvector $X_1$, show that the matrix $B = A - \lambda_1 X_1 X_1^T$ can be used to obtain, using the power method, the eigenvector $X_2$. [Assume $|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_n| > 0$]. What would you do (which matrix would you use and why) if you were asked to find, instead of $X_2$, the eigenvector $X_3$?

3. Oftentimes, it is possible to improve Gerschgorin’s estimates, by using a diagonal matrix as follows:
   - Consider instead of matrix $A$, the similar matrix $D^{-1}AD$, where $D = \text{diag}(d_1, \ldots, d_n)$. Show then that the Gerschgorin circles $C_i$ are given by $|\lambda - a_{ii}| \leq \sum_{k=1, k \neq i}^n \left| \frac{a_{ik}d_k}{d_i} \right|$.
   - Consider the matrix $A = \begin{bmatrix} 1, \epsilon, \epsilon, 2, \epsilon, \epsilon, \epsilon, 2 \end{bmatrix}$. Use $B = D^{-1}AD$ to find the new Gerschgorin circles for $D = \text{diag}(1, k\epsilon, k\epsilon)$. Assume that $k > 0$ and $\epsilon > 0$.
   - If $\rho_1$ and $\rho_2 = \rho_3$ are the two Gerschgorin radii for $B$, find the $k$ that for a fixed $\epsilon$ minimizes the sum $\rho = \rho_1 + \rho_2$. Calculate the same quantity for the original matrix $A$ and show that indeed $\rho$ (which is a measure of the uncertainty in knowing the eigenvalues) is indeed smaller for $B$. 