Homework Problem Set 2

1. Consider the problem \( A \cdot x = c \) with 
\[
A = \begin{bmatrix} 3 & 4 & 5; 5 & 2 & 3; 6 & 3 & 7 \end{bmatrix}; \quad c = \begin{bmatrix} 7, 3, 4 \end{bmatrix}'.
Following what we discussed in class, give a Matlab code that solves this linear system using Gauss elimination with partial pivoting. To debug your code, you should check that the answer that you get is the same answer for \( x \approx A \cdot c \) and furthermore your final \( A \) should be the same as \( B = lu([3, 4, 5; 2, 3; 6, 3, 7]) \).

2. Condition Numbers and Errors:

- For an orthogonal matrix \( U \) (i.e., one such that \( U^T U = I \)), show that the condition number with respect to the 2-norm, \( \text{cond}(U) = 1 \).
- Solve problem 18 of Chapter 8 in Atkinson. As a remark on the moral of this problem, notice that this clearly illustrates a specific case where the residual can be small but the relative error is of order 1 (because of the large condition number). It is also a case that essentially saturates the right hand side of the estimate of (8.4.4).
- Suppose we decompose the matrix \( A = D + L + U \), where \( D \) is the diagonal part, \( L \) is the lower triangular part of \( A \) (zeros at the diagonal and above) and \( U \) the upper triangular part of \( A \) (zeros at the diagonal and below). Consider \( B = D + L \). If \( d_1 = \max_i |a_{ii}| \) and \( d_2 = \min_i |a_{ii}| \), express \( \text{cond}(B) \) in terms of \( d_1 \) and \( d_2 \).

3. Iterative Schemes: Consider their general form \( x^{k+1} = M \cdot x^k + b \).

- Show that the iteration matrix \( M \) is singular in the case of the Gauss-Seidel method.
- Show that if \( ||M|| < 1 \), then
\[
||x^{k+1}|| \leq ||M||^k ||x^0|| + \frac{1}{1 - ||M||} ||b||
\]
- Considering the absolute error \( ||x^{k+1} - x|| \) (where \( x \) is the solution of the above iterative scheme), show directly that if \( ||M|| < 1 \), then the iterative scheme converges.