NOTES FOR MATH 132

Antiderivatives

Definition: A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$ ($\forall x \in I$).

Recall that antiderivatives are not unique, and the following theorem gives the connection between all antiderivatives of a function.

Theorem: If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is $F(x) + C$, where $C$ is an arbitrary constant ($\forall C \in \mathbb{R}$).

This most general antiderivative is also called an indefinite integral of $f$ and is denoted by $\int f(x)dx$.

Areas

To define the area under a continuous curve $y = f(x)$ over an interval $[a, b]$ the following steps are needed:

1. divide the interval $[a, b]$ into $n$ equal (having equal width) subintervals $[x_{i-1}, x_i]$, $i = 1, 2, ..., n$ with endpoints $x_0, x_1, ..., x_n$

   $a = x_0 < x_1 < ... < x_n = b$.

2. construct approximating rectangles having as bases the intervals $[x_{i-1}, x_i]$, and as heights $f(x_i)$.

3. improve the approximation by increasing $n$, and eventually passing to the limit $n \to \infty$.

Definition: The area $A$ of the region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[ f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right].$$

Note: The limit in the definition is guaranteed to exist provided the function is continuous.
**Definite Integral**

$f$ is a continuous function on the interval $[a, b]$:

1. divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$ with endpoints $x_0, x_1, ..., x_n$:

   $a = x_0 < x_1 < ... < x_n = b.$

2. choose sample points $x_1^*, x_2^*, ..., x_n^*$ in these subintervals ($x_i^* \in [x_{i-1}, x_i]$).

3. $\sum_{i=1}^{n} f(x_i^*) \Delta x$ is called Riemann sum for the function $f$ over interval $[a, b]$.

**Definition:** The definite integral of $f$ from $a$ to $b$ is defined to be

\[
\int_{a}^{b} f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

in this notation $a$ and $b$ are called limits of integration ($a$-lower limit, $b$-upper limit); $f$ is called integrand.

**Note:** The limit in the definition is guaranteed to exist provided the function is continuous.

Notice that with the definition of Definite Integral the area under the curve $y = f(x)$ over the interval $[a, b]$ is exactly $\int_{a}^{b} f(x)dx$.

**Properties of Definite Integrals**

1. $\int_{a}^{b} f(x)dx = -\int_{a}^{b} f(x)dx$.

2. $\int_{a}^{a} f(x)dx = 0$.

3. $\int_{a}^{b} cdx = c(b - a)$ $c$ is any constant $\forall c \in \mathbb{R}.$

4. $\int_{a}^{b} [f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$.

5. $\int_{a}^{b} cf(x)dx = c \int_{a}^{b} f(x)dx$ $c$ is any constant $\forall c \in \mathbb{R}.$

6. $\int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx = \int_{a}^{b} f(x)dx$.

Comparison properties

7. if $f(x) \geq 0$ for $\forall x \in [a, b] \Rightarrow \int_{a}^{b} f(x)dx \geq 0$.

8. if $f(x) \leq g(x)$ for $\forall x \in [a, b] \Rightarrow \int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx$.

9. if $m \leq f(x) \leq M$ for $\forall x \in [a, b] \Rightarrow m(b - a) \leq \int_{a}^{b} f(x)dx \leq M(b - a)$. 

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\textbf{Note:} \( \int_a^b f(x)g(x)dx \neq \left( \int_a^b f(x)dx \right) \left( \int_a^b g(x)dx \right). \)

\textbf{Fundamental Theorem of Calculus}

\textbf{FTC Part 1:} If \( f \) is continuous on \([a,b]\) then the function defined by \( g(x) = \int_a^x f(t)dt, \quad a \leq x \leq b; \) is continuous on \([a,b]\), is differentiable on \((a,b)\) and \( g'(x) = f(x) \) (\( g \) is an antiderivative of \( f \)).

\textbf{FTC Part 2:} If \( f \) is continuous on \([a,b]\), then \( \int_a^b f(x)dx = F(b) - F(a) \) where \( F \) is any antiderivative of \( f \).

Notice that \textbf{FTC} suggests that differentiation and integration are inverse operations:

\[
\frac{d}{dx} \left[ \int_a^x f(t)dt \right] = f(x)
\]

\[
\int F'(x)dx = F(x).
\]

\textbf{Techniques of Integration}

\textbf{Substitution Rule}

For indefinite integrals:
If \( u = g(x) \) is a differentiable function having range on interval \( I \) and \( f \) is continuous on \( I \) then

\[
\int f(g(x))g'(x)dx = \int f(u)du.
\]

For definite integrals:
If \( g \) is continuous on \([a,b]\) and \( f \) is continuous on the range of \( u = g(x) \) then

\[
\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.
\]

\textbf{Note:} it is permissible to operate with \( dx \) and \( du \) after the integral signs as if they were differentials \((dg(x) = g'(x)dx \) or \( du = \frac{du}{dx}dx; \quad \frac{du}{dx} = \text{derivative}\)).

\textbf{Integration by Parts}

For indefinite integrals:

\[
\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx
\]
or equivalently taking \( u = f(x), v = g(x); \) \( \int u dv = uv - \int v du. \)

For definite integrals:

\[
\int_a^b f(x)g'(x)dx = f(x)g(x) \bigg|_a^b - \int_a^b g(x)f'(x)dx.
\]

**Trigonometric Integrals**

Strategy for evaluating \( \int \sin^m(x) \cos^n(x)dx \):

a) if \( n = 2k + 1 \) (the power of cosine is odd)

\[
\int \sin^m(x) \cos^n(x)dx = \int \sin^m(x)(1 - \sin^2(x))^k \cos(x)dx
\]

make substitution \( u = \sin x \).

b) if \( m = 2k + 1 \) (the power of sine is odd)

\[
\int \sin^m(x) \cos^n(x)dx = \int (1 - \cos^2(x))^k \cos^n(x) \sin(x)dx
\]

make substitution \( u = \cos x \).

c) if both \( n \) and \( m \) are even, then use power reduction formulas (half-angle identities):

\[
\sin^2 x = \frac{1}{2} (1 - \cos(2x)) \quad \cos^2 x = \frac{1}{2} (1 + \cos(2x)).
\]

Strategy for evaluating \( \int \tan^m(x) \sec^n(x)dx \):

a) if \( n = 2k \) (the power of secant is even)

\[
\int \tan^m(x) \sec^n(x)dx = \int \tan^m(x)(1 + \tan^2(x))^{k-1} \sec^2(x)dx
\]

make substitution \( u = \sec x \).

b) if \( m = 2k + 1 \) (the power of tangent is odd)

\[
\int \tan^m(x) \sec^n(x)dx = \int (\sec^2(x) - 1)^k \sec^{n-1}(x) \tan(x) \sec(x)dx
\]

make substitution \( u = \tan x \).

c) if \( n \) is odd and \( m \) is even - no standard methods.
Strategy for a) $\int \sin(mx) \cos(nx) \, dx$, b) $\int \sin(mx) \sin(nx) \, dx$, c) $\int \cos(mx) \cos(nx) \, dx$:

a) $\sin A \cos B = \frac{1}{2} \sin(A - B) + \sin(A + B)$.

b) $\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$.

c) $\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$.

**Trigonometric Substitution**

Use the following trigonometric substitutions for the corresponding expressions:

- $\sqrt{a^2 - x^2} \quad x = a \sin \theta; \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad (a^2 - a^2 \sin^2(x) = a^2 \cos^2(x))$
- $\sqrt{a^2 + x^2} \quad x = a \tan \theta; \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad (a^2 + a^2 \tan^2(x) = a^2 \sec^2(x))$
- $\sqrt{x^2 - a^2} \quad x = a \sec \theta; \quad 0 \leq \theta < \frac{\pi}{2} \quad (a^2 \sec^2(x) - a^2 = a^2 \tan^2(x))$

**Improper Integrals**

**Improper Integrals of Type 1:**

**Definition:**

a) If $\int_a^t f(x) \, dx$ exists for every number $t \geq a$, then define $\int_a^\infty f(x) \, dx =$

$$\lim_{t \to \infty} \int_a^t f(x) \, dx,$$

provided the limit exists (as a finite number).

b) If $\int_b^t f(x) \, dx$ exists for every number $t \leq b$, then define $\int_\infty^b f(x) \, dx =$

$$\lim_{t \to -\infty} \int_t^b f(x) \, dx,$$

provided the limit exists (as a finite number).

c) if $\int_a^\infty f(x) \, dx$ and $\int_\infty^a f(x) \, dx$ are both defined (corresponding limits exist as finite numbers), then we define $\int_\infty^\infty f(x) \, dx =$ $\int_a^\infty f(x) \, dx + \int_\infty^a f(x) \, dx$

(any number $a$ can be used).

In these definitions the improper integrals are called *convergent* if the corresponding limits exist (as finite numbers), and *divergent* otherwise.

*Note:* for c) both integrals in a) and b) must be convergent.
Improper Integrals of Type 2:

**Definition:**

a) If \( f \) is continuous on \([a, b]\) and has a (infinite) discontinuity at \( b \) then define \( \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx \), provided the limit exists (as a finite number).

b) If \( f \) is continuous on \((a, b]\) and has a (infinite) discontinuity at \( a \) then define \( \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx \), provided the limit exists (as a finite number).

c) If \( f \) has a (infinite) discontinuity at \( c \) where \( a < c < b \), and both \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) are defined (corresponding limits exist as finite numbers), then we define \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \).

In these definitions the improper integrals are called *convergent* if the corresponding limits exist (as finite numbers), and *divergent* otherwise.

**Note:** for c) both integrals in a) and b) must be convergent.

**Curves Defined by Parametric Equations**

**Tangents**

If the curve \( y = F(x), \ a \leq x \leq b \) is also given by parametric equations

\[
\begin{align*}
&\begin{align*}
  x &= f(t), & &a \leq t \leq \beta; \\
y &= g(t)
\end{align*}
\end{align*}
\]

then the following formula for the tangent to the curve in terms of the parametric equations holds:

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0 \quad \text{or} \quad F'(x) = \frac{g'(t)}{f'(t)} \quad \text{if} \quad f'(t) \neq 0.
\]

- If \( \frac{dy}{dt} = 0 \) and \( \frac{dx}{dt} \neq 0 \) \( \rightarrow \) horizontal tangent.
- If \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \) \( \rightarrow \) vertical tangent.

(The case \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} = 0 \) must be handled with more care considering \( \lim(\frac{dy}{dx})/(\frac{dx}{dt})\)).

Similarly for the second derivative one has:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dt} \right) \frac{d}{dx} \left( \frac{dx}{dt} \right).
\]

**Note:**

\[
\frac{d^2y}{dx^2} \neq \frac{d^2y}{dt^2} \frac{dx}{dt}.
\]
Areas
If the curve $y = F(x), \quad a \leq x \leq b$ is also given by parametric equations
\[
\begin{aligned}
x &= f(t) \\
y &= g(t),
\end{aligned}
\quad \alpha \leq t \leq \beta;
\] and $a = f(\alpha), b = f(\beta)$, then the area under the curve over the interval $[a, b]$ is:
\[
A = \int_{a}^{b} F(x)dx = \int_{\alpha}^{\beta} g(t)f'(t)dt.
\]

Arc Length
Similar to areas under curves, one can proceed with the definition of arc length of a curve by dividing the interval on which the curve is defined into subintervals, and taking the limit of the lengths of approximating polygons, having vertices at the points on the curve corresponding to the endpoints of subintervals.

If the curve $C$ is given by its parametric equations
\[
\begin{aligned}
x &= f(t) \\
y &= g(t)
\end{aligned}
\quad \alpha \leq t \leq \beta;
\] both $f'$ and $g'$ are continuous on $[\alpha, \beta]$ (and $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$) then the length of $C$ in terms of parametric equations is given by:
\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.
\]

Polar Coordinates
Change of coordinates from polar coordinates to cartesian coordinates $[(r, \theta) \rightarrow (x, y)]$ is given by the following formulas:
\[
\begin{aligned}
x &= r \cos \theta \\
y &= r \sin \theta
\end{aligned}
\]
Change of coordinates from cartesian coordinates to polar coordinates \([(x, y) \rightarrow (r, \theta)]\) is given by the following formulas:

\[
\begin{align*}
 r^2 &= x^2 + y^2 \\
 \tan \theta &= \frac{y}{x} 
\end{align*}
\]

**Tangents to polar curves**

If the curve \(y = F(x)\) is also given in terms of polar coordinates \(r = f(\theta)\), then the tangent to the curve (in terms of polar coordinates) is given by:

\[
\frac{dy}{dx} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \quad \text{or} \quad F'(x) = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.
\]

**Areas of polar regions**

The same process of approximation is employed in the definition of the area of polar regions, with the only difference of taking circular segments instead of approximating rectangles.

The area of a polar region \(R\), bounded by the curve \(r = f(\theta)\) and rays \(\theta = a\) and \(\theta = b\) is given by:

\[
A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta \quad \text{or} \quad A = \int_a^b \frac{1}{2} r^2 d\theta.
\]

**Arc Length**

By treating \(\theta\) as a parameter in the formula for the arc length, one easily finds that the length of a polar curve \(r = f(\theta)\) with \(a \leq \theta \leq b\) is:

\[
L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad \text{or} \quad L = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta.
\]