Stat705 Exam 1

This exam is closed book and closed notes. All answers are worth an equal number of points.

1. Let $y_i = \mu + e_i, e_i \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2), i = 1, \ldots, n$.

   (a) Write a matrix expression for $\widehat{\mu}$, the least squares estimator of μ.

   (b) Consider $r_i = y_i - \widehat{y}_i, i = 1, \ldots, n$. (Note: $\widehat{y}_i = \widehat{\mu}$).

      i. What is the distribution of $\mathbf{r} = (r_1, \ldots, r_n)^{\mathrm{T}}$?

      ii. Are r_1 and r_2 independent? Why or why not?

      iii. What is the expected value of the estimated variance of r_i? (i.e. $\widehat{var}(r_i) = \frac{1}{n}\sum_{i=1}^{n}(r_i - \overline{r})^2$ where $\overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$.) Please use a quadratic form to get the result. (Hint to simplify: What does $\mathbf{1}^{\mathrm{T}}\mathbf{r}$ equal?)

2. Suppose $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}), \mathbf{X}$ is n by p, rank $p < n$.

   (a) Write down the pdf of $\mathbf{y}$.

   (b) Let $\mathbf{U}$ be an n by n orthonormal matrix ($\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}_n$). Let $\mathbf{z} = \mathbf{U}\mathbf{y}$.

      i. What is the distribution of $\mathbf{z}$?

      ii. Suppose $\mathbf{z}$ is observed and $\mathbf{X}$ and $\mathbf{U}$ are known. What is the MLE of $\boldsymbol{\beta}$?

3. Let $\mathbf{X} = (\mathbf{1}_n \quad \mathbf{x} \quad \mathbf{x}^2 \quad \mathbf{x}^3)$ and $\tilde{\mathbf{X}} = (\mathbf{1}_n \quad \mathbf{x})$ where $\mathbf{x}^k = (x_1^k, \ldots, x_n^k)^{\mathrm{T}})$. Let $\mathbf{P} = \mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}$ and $\tilde{\mathbf{P}} = \tilde{\mathbf{X}}(\tilde{\mathbf{X}}^{\mathrm{T}}\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}^{\mathrm{T}}$. Assume the inverses exist. Let $F = (\mathbf{y}^{\mathrm{T}}(\mathbf{P} - \tilde{\mathbf{P}})\mathbf{y}/2)/(\mathbf{y}^{\mathrm{T}}(\mathbf{I}_n - \mathbf{P})\mathbf{y}/(n-4))$.

   (a) What two models are compared by F?

   (b) Suppose F = 100,000. Which model would you *probably* prefer and why?

   (c) Let $\mathbf{y} \sim N(\tilde{\mathbf{X}}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n)$. Are the numerator and denominator of F independent? Why or why not?