1 5.2: Note this is a question about conditional distributions.
   (a) With \( X_1 \) known, \( S(x) = 1 - F(x) = Pr(X_i \geq x | X_1 = x), i > 1 \). Integrate out the condition to get \( Pr(X_i > X_1) = \int_0^\infty S(x)f(x)dx \). Without knowing the form of \( f(x) \), we can’t make this simpler.
   (b) Let \( N \) be years until rainfall exceeds \( X_1 \). With \( X_1 = x \) known, \( N \) is geometric with \( p = S(x) = 1 - F(x) \). \( E(N|X_1 = x) = 1/S(x) \) and \( E(N) = E_X(E(N|X)) = E_X(1/S(X)) = \int_0^\infty f(x)/S(x)dx = -\log(S(x))|_0^\infty = \infty \). Note: The derivative of \( S(x) \) is \( -f(x) \).

2 5.3. The \( Pr(Y_i = 1) = Pr(X_i > \mu) = 1 - F_X(\mu) \) and \( Pr(Y_i = 0) = F_X(\mu) \). Since the \( X_i \) are independent, the \( Y_i \)s are too. See Theorem 4.6.12 on page 184. As a result, the \( Y_i \)s are iid Bernouilli \( \{1 - F_X(\mu)\} \), and their sum is binomial.

3 5.5: Let \( Y = X_1 + \ldots + X_n \) and \( \bar{X} = Y/n \) and \( n\bar{X} = Y \). The result follows from the definition of a univariate transformation:
\[
f_{\bar{X}}(\bar{x}) = f_Y(n\bar{x})d(n\bar{x})/d\bar{x} = f_Y(n\bar{x})n
\]
Note that \( \bar{x} \) is just a dummy argument. Using \( x \) instead doesn’t change anything.

4 5.6:
   (a) Let \( Z = X - Y, W = X \). The inverse is \( X = W \) and \( Y = W - Z \). The Jacobian is 1 and \( f_{Z,W}(z,w) = f_X(w)f_Y(w-z) \). As a result, \( f_Z(z) = \int_w f_Y(w-z)f_X(w)dw \).
   (b) Let \( W = X, Z = XY \). The inverse is \( X = W \) and \( Y = Z/W \). The absolute value of the Jacobian is \( z/w^2 \), and \( f_{Z,W}(z,w) = f_X(w)f_Y(z/w)z/w^2 \). As a result, \( f_Z(z) = \int_w z/w^2 f_Y(z/w)f_X(w)dw \).
   (c) Let \( W = Y, Z = X/Y \). The inverse is \( X = ZW \) and \( Y = W \). The absolute value of the Jacobian is \( w^2 \), and \( f_{Z,W}(z,w) = f_X(zw)f_Y(w)w^2 \). As a result, \( f_Z(z) = \int_w w^2 f_X(zw)f_Y(w)dw \).

5 5.8 OK. I haven’t found a way to do this that isn’t extremely long and tedious. Sorry!

6 5.10
   (a) Start with \( E(X_i) = \mu, E(X_i^2) = \sigma^2 + \mu^2 \). As a result, \( \theta_1 = \mu \) and \( \theta_2 = \sigma^2 \). Derivatives of the moment generating function \( \exp(t\mu + t^2\sigma^2/2) \) evaluated at \( t = 0 \) can be used to get \( E(X_i^3) = \mu^3 + 3\mu\sigma^2 \) and \( E(X_i^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \). After this, algebra shows that \( \theta_3 = E(X_i - \mu)^3 = 0 \) and \( \theta_3 = E(X_i - \mu)^4 = 3\sigma^4 \).
   (b) Plugging in shows that \( var(S^2) = (3\sigma^4 - \frac{n-3}{n-1}\sigma^4)/n \).
   (c) Since the variance of a \( \chi^2_{n-1} \) random variable is \( 2(n-1) \), the variance of \( (n-1)S^2/\sigma^2 \) is \( 2(n-1) \). Since \( S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1} \), the variance of \( S^2 \) is \( \frac{\sigma^2(n-1)}{(n-1)(n-1)} = \frac{2\sigma^4}{n-1} \). We’ve used that the variance of \( cX \) is \( c^2 var(X) \). Note that
\[
(3\sigma^4 - \frac{n-3}{n-1}\sigma^4)/n = \frac{3\sigma^4}{n} - \frac{(n-3)\sigma^4}{n(n-1)} = \frac{3(n-1) - n + 3}{n(n-1)}\sigma^4 = 2\sigma^4/(n-1)
\]
7 5.11: Use Jensen’s Inequality to show that \( E(S) \leq \sigma \). Equality only occurs if \( Pr(S = S^2) = 1 \) which is equivalent to \( Pr(S = 0) = 1 \) or \( Pr(S = 1) = 1 \). Assuming \( X_i \) is continuous, \( Pr(S = 1) = 0 \). The event \( S = 0 \) only occurs if \( X_i = X \) for all \( i \). That can only happen if \( Pr(X_i = c) = 1 \) for all \( i \). If \( \sigma > 0 \), then that won’t happen.

8 5.13: (Assume we can’t use the trivial solution \( S^2 / \sigma \).) Let \( X = (n - 1)S^2 / \sigma^2 \). We know \( X \sim \chi^2_{n-1} \) and \( S = \sqrt{S^2} = \frac{\sigma X^{1/2}}{\sqrt{n-1}} \). Using the hint, let’s try to find a \( c \) so that \( E(cS) = E\left(c\frac{\sigma X^{1/2}}{\sqrt{n-1}}\right) = \sigma \). Using the \( \chi^2_{n-1} \) pdf,

\[
E\left(c\frac{\sigma X^{1/2}}{\sqrt{n-1}}\right) = \int_0^\infty \frac{c\sigma x^{1/2}x^{(n-1)/2-1} \exp(-x/2)}{\sqrt{n-1} \Gamma((n-1)/2)2^{(n-1)/2}} \, dx
\]

\[
= \sigma \frac{c}{\sqrt{n-1} \Gamma((n-1)/2)2^{(n-1)/2}} \int_0^\infty x^{n/2-1} \exp(-x/2) \, dx.
\]

Recognizing a kernel for the \( \chi^2_n \) distribution, we get

\[
E\left(c\frac{\sigma X^{1/2}}{\sqrt{n-1}}\right) = \sigma \frac{c \Gamma(n/2)2^{n/2}}{\sqrt{n-1} \Gamma((n-1)/2)2^{(n-1)/2}}.
\]

As a result, letting \( c = \left(\frac{\Gamma(n/2)2^{n/2}}{\sqrt{n-1} \Gamma((n-1)/2)2^{(n-1)/2}}\right)^{-1} \) gives \( E\left(c\frac{\sigma X^{1/2}}{\sqrt{n-1}}\right) = \sigma \).