Solution to problem set three. (Stat607, Fall 2005)

1. 2.30

(a)

\[ M(t) = \int_0^a \exp(tx)/adx \]
\[ = (\exp(ta) - 1)/(ta). \]

(b)

\[ M(t) = \int_0^c \exp(tx)2x/c^2 \]
\[ = 2/c^2 \int_0^c x \exp(tx) \]
\[ = 2/(tc^2) \left( (xe^{tx}) \bigg|_0^c - \int_0^c \exp(tx) dx \right) \]
\[ = \frac{2}{c^2 t} \left( c \exp(ct) - \frac{\exp(ct) - 1}{t} \right). \]

L’Hopital can be used to show that it is finite for \( t \) in n-hood of zero.

(c)

\[ M(t) = \int_{-\infty}^{\infty} \exp(tx) \exp(-|x-\alpha|/\beta)/(2\beta) dx \]
\[ = \frac{1}{2\beta} \left[ \int_{-\infty}^{a} \exp(tx) \exp \left( \frac{x-\alpha}{\beta} \right) dx + \int_{a}^{\infty} \exp(tx) \exp \left( \frac{\alpha-x}{\beta} \right) dx \right] \]
\[ = \frac{\exp(at) - 1}{\beta} \quad -1/\beta < t < 1/\beta. \]

(used exponential kernel)

(d)

\[ M(t) = \sum_{x=0}^{\infty} \exp(tx) \left( \frac{r+x-1}{x} \right) p^r (1-p)^x \]
\[ = p^r \sum_{x=0}^{\infty} \left( \frac{r+x-1}{x} \right) \{\exp(t)(1-p)\}^x \]
\[ = p^r/[1 - \exp(t)(1-p)]^r, \quad t \leq -\log(1-p). \]

The last line treats \( \exp(t)(1-p) \) like \( q \). When this \( q \) is between zero and 0, then \( \sum_{x=0}^{\infty} \left( \frac{r+x-1}{x} \right) (1-q)^r q^x = 1 \) or \( \sum_{x=0}^{\infty} \left( \frac{r+x-1}{x} \right) q^x = (1-q)^{-r}. \)
2. 2.32 See page 83 in text for a brief discussion of the cumulants. Since 
\( S(t) = \log(M(t)), \frac{d}{dt} S(t) = M'(t)/M(t) \). Note that \( M(0) = 1 \) since a density integrates or sums to one. This shows that \( \frac{d}{dt} S(t)|_{t=0} = EX \). Next, 
\( \frac{d^2}{dt^2} S(t) = M''(t)/M(t) - M'(t)M'(t)/M(t)^2 \). The result follows.

3. 3.2.

(a) Let
\[ A = \text{event that the manufacturer accepts an unacceptable lot.} \]
\[ = \geq 6 \text{ defectives and not detected in sample of size } K. \]
We want a (minimal) \( K \) so that \( Pr(A) < 0.10 \). Suppose \( M \) parts are defective out of \( N \) total, and a sample of size \( K \) is chosen. Let \( X_M \) be a random variable that represents the number of defectives in the sample. \( X_M \sim Hyper(100, M, K) \). As a result, if \( M \) were known, \( Pr(A) = Pr(X_M = 0) \). By differentiating the probability mass function, we can show that \( Pr(X_M = 0) \geq Pr(X_{M+1} = 0) \). (This makes intuitive sense since \( M \) is the number of defectives out of 100.) As a result, we only need to find a \( K \) so that \( Pr(X_6 = 0) < 0.10 \) (since \( Pr(X_6 = 0) \geq Pr(X_7 = 0) \geq \ldots \)). Trial and error is a straightforward way to find such a \( K \), and \( K = 33 \) suffices.

(b) By similar reasoning to the above, we want a \( K \) so that \( Pr(X_6 = 0) + Pr(X_6 = 1) \leq 0.1 \). Now \( K = 54 \) is sufficient. (Note: Since this set up makes it easier to accept a faulty lot mistakenly, it makes sense that \( K \) increases.)

4. 3.3 Let \( X_t \) be 0 if a car doesn’t pass at time \( t \) and 1 if one does. \( X_t \) are \( iid \) Bern(\( p \)). I assume the person is at the corner at time 0. (Alternatively, one could also start the clock at time 1 and get a slightly different answer.) Since the person couldn’t cross at time 0, \( X_1 + X_2 + X_3 > 0 \). Similarly, since the person couldn’t cross at time 1, \( X_4 \) must be 1. Next, \( X_5 + X_6 + X_7 = 0 \) since the person can cross at time 4. Using independence of the \( X_t \)s,
\[ Pr(X_1 + X_2 + X_3 > 0) = 1 - (1 - p)^3 \]
\[ Pr(X_4 = 1) = p \]
\[ Pr(X_5 + X_6 + X_7 = 0) = (1 - p)^3. \]
Using independence again, the probability of the whole sequence is \( [1 - (1 - p)^3]p(1 - p)^3. \)

5. 3.4

(a) Letting \( X \) be number of tries until success, \( X \) has a geometric distribution with \( p = 1/n \) since one out of \( n \) keys is the correct one. As a result, \( E(X) = n. \)
(b) In this case, let’s derive the pmf for $X$ as defined above. Let $Y_n = 1$ be the event that he finds the key on the $n$th try and $Y_n = 0$ be the event that he does not. $Pr(X = 1) = Pr(Y_1 = 1) = 1/n$ since there is one correct key out of $n$. $Pr(X = 2) = Pr(Y_1 = 0, Y_2 = 1) = ((n - 1)/n)(1/(n - 1)) = 1/n$ by independence. Similarly, $Pr(X = 3) = Pr(Y_1 = 0, Y_2 = 0, Y_3 = 1) = ((n - 1)/n)((n - 2)/(n - 1))(1/(n - 2)) = 1/n$. A bit surprisingly, (to me anyway) one can use induction to show that $Pr(Y_1 = 0, \ldots, Y_{k-1} = 0, Y_k = 1) = 1/n$. As a result, $E(X) = \sum_{i=1}^{n}(i/n) = n(n + 1)/(2n) = (n + 1)/2$.

6. 3.9 Define class as all 60 students.

(a) Let $X$ be the number of pairs of children in the class that are twins. $X \sim Bin(30, 1/90)$. (This seems better than the hint in the book, but this model doesn’t seem too great either, frankly.) The event ($X \geq 10$) describes the observed event or something more extreme. The probability of this is about 1.9 times $10^{-5}$. That seems fairly rare, but not “impossible”.

(b) Let $Y$ be the number of classes in the state that have five sets of twins in them. Using the result from the previous problem, $Y \sim Bin(30 \times 62 \times 5, 0.000019)$. We want $1 - Pr(Y = 0) = 0.16$. That is relatively likely.

(c) Let $Z$ be the number of classes in the state that have five sets of twins in them. Using the result from part a, $Z \sim Bin(310 \times 50 \times 10, 0.000019)$. We want $1 - Pr(Z = 0) \approx 1$. That is quite likely.

7. 3.19 The integration by parts was started in Example 3.3.1. If $X \sim gamma(\alpha, \beta)$ and $Y \sim Poisson(x/\beta)$ then the relationship is $Pr(X \leq x) = Pr(Y \geq \alpha)$. See notes for discussion of what this means.