Solution to problem set two. (Stat607, Fall 2004)

1. 1.34 Let the events $L_1, L_2$ denote which litter was chosen, and the event $B$ denote choosing a brown haired animal.

(a) 
\[ Pr(B) = Pr(B|A_1)Pr(A_1) + Pr(B|A_2)Pr(A_2) \]
\[ = (2/3)(1/2) + (3/5)(1/2) = 19/30. \]

(b) 
\[ Pr(A_1|B) = Pr(B, A_1)/Pr(B) \]
\[ = Pr(B|A_1)Pr(A_1)/Pr(B) \]
\[ = (2/3)(1/2)/(19/30) = 10/19. \]

2. 1.38

(a) By Bonferroni, $Pr(B \cap A) \geq Pr(A) + Pr(B) - 1 = Pr(A).$ By Theorem 1.2.9, since $B \cap A \subseteq A, Pr(B \cap A) \leq Pr(A).$ Hence, $Pr(B \cap A) = Pr(A).$ By Bayes rule, $Pr(A|B) = Pr(A \cap B)/Pr(B) = Pr(A).$ Since $B$ has probability 1, every event is independent of $B.$

(b) Since $A \subset B, A \cap B = A.$ Use this in combination with Bayes rule to get the desired results.

(c) Mutually exclusive means $A \cap B = \emptyset$ and $Pr(A \cap B) = 0.$ Using Bayes rule,
\[ Pr(A|A \cup B) = Pr(A \cap (A \cup B))/Pr(A \cup B) \]
\[ = Pr(A \cup (A \cap B))/Pr(A \cup B). \]

(d) $Pr(A \cap B \cap C) = Pr(A|B \cap C)Pr(B \cap C)$ by Bayes rule. Use Bayes rule again on $Pr(B \cap C).$

3. 1.55 The goal is to build the function $F_V(v) = Pr(V < v)$ for all $v.$

- First, since $T < 3$ results in $V = 5, Pr(V < v) = 0$ for all $v < 5.$
- Next, since $V \geq 6$ only occurs when $T \geq 3, Pr(V < v) = Pr(T < 3)$ for $5 \leq 3 < 6$ with $Pr(T < 3) = \int_0^3 (1/1.5) \exp(-x/1.5)dx = 1 - \exp(-2).$
- Finally, for $v \geq 6, P(V < v)$ is identical to $Pr(2T < v) = Pr(T < v/2) = \int_0^{v/2} (1/1.5) \exp(-x/1.5)dx = 1 - \exp(-v/3).$

4. 2.14a
\[ E(X) = \int_0^\infty xf(x)dx \]
\[
\lim_{a \to \infty} \left\{ xF(x)\bigg|_0^a - \int_0^a F(x)\,dx \right\} \\
= \lim_{a \to \infty} \left\{ aF(a) - \int_0^a F(x)\,dx \right\} \\
= \lim_{a \to \infty} \left\{ F(a) \int_0^a \,dx - \int_0^a F(x)\,dx \right\} \\
= \lim_{a \to \infty} \left\{ F(a) \int_0^a \left(1 - \frac{F(x)}{F(a)}\right)\,dx \right\}.
\]

Since \( F(a) \) goes to 1, this is
\[
\int_0^\infty (1 - F(x))\,dx.
\]

Note that the previous derivation does not show that \( E(X) \) exists or not. If we start in the “other direction” and integrate \( \int_0^a (1 - F(x))\,dx \) by parts, then we can see that the integral is finite if \( \lim_{a \to \infty} a(1 - F(a)) \) is finite.

5. 2.17

(a) Want an \( m \) such that \( \int_0^m f(x)\,dx = 0.5 \). Doing the integration gives \( x^3\bigg|_0^0.5 = 0.5 \) or \( m = \text{cube root of } 0.5 \).

(b) Similar to the last problem, we get \( \pi^{-1} \arctan x\bigg|_{-\infty}^\infty = 0.5 \). Solving for \( m \) gives \( m = 0 \). See Example 2.2.4 for a calculation that shows that the mean of the Cauchy does not exist. The median always exists.

6. 2.18 By the definition of absolute value, \( E|X - a| = \int_{-\infty}^a (a - x)\,f(x)\,dx + \int_a^\infty (x - a)\,f(x)\,dx \). Assume that we can exchange the order of integration. Dominated convergence can be used to show that we can here. By Liebnitz’s rule: \( d/daE|X - a| = \int_{-\infty}^a f(x)\,dx - \int_a^\infty f(x)\,dx \). By definitions of CDFs and the median, is equals zero when \( a = m \). Similarly, \( d^2/da^2E|X - a| = 2f(a) \geq 0 \). Since \( F(m) = 1/2 = \int_{-\infty}^m f(x)\,dx \) for a continuous random variable, there needs to be some point \( m \) where \( f(m) > 0 \).

7. 2.26

(a) Examples of symmetric PDFs: normal, Cauchy, uniform, double exponential...

(b) \( \int_{-\infty}^\infty f(x)\,dx = \int_0^\infty f(a + y)\,dy = \int_0^\infty f(a - y)\,dy = \int_{-\infty}^a f(x)\,dx \). Since \( \int_0^\infty f(a + y)\,dy \) and \( \int_{-\infty}^a f(x)\,dx \) also add to one, \( a \) is the median.

(c)
\[
EX = \int_{-\infty}^a xf(x)\,dx + \int_a^\infty xf(x)\,dx.
\]
Let $y = x - a$, $x = y + a$.

\[
EX = \int_{-\infty}^{0} (y + a)f(a + y)dy + \int_{0}^{\infty} (a + y)f(a + y)dy \\
= a\int_{-\infty}^{0} f(a + y)dy + \int_{0}^{\infty} f(a + y)dy \\
+ \int_{-\infty}^{0} yf(a + y)dy + \int_{0}^{\infty} yf(a + y)dy \\
= a + \int_{-\infty}^{0} yf(a - y)dy + \int_{0}^{\infty} yf(a + y)dy \\
= a + \int_{0}^{\infty} (-y)f(a + y)dy + \int_{0}^{\infty} yf(a + y)dy \\
= a.
\]

(d) $e^{-(x+d)} \neq e^{-(x+d)}$ for any $d > 0$.

(e) Integration shows that the mean is 1, and the median is $\log(2)$.

8. It’s perhaps a bit tedious, but the MGFs can be derived for both distributions. Evaluating the first five derivatives of each at 0 gives the result.

9. Let $N$ be the number of flips until a head occurs. (Note that $N$ has a geometric distribution with parameter $p$.) Since the first head on the $n$th flip must have been preceded by $n - 1$ tails, $Pr(N = n) = (1 - p)^{n-1}p$. $E(N) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{i-1}$. Since $s(x) = \sum_{i=1}^{\infty} x^i = x/(1-x), s'f(x) = \sum_{i=1}^{\infty} ix^{i-1} = 1/(1-x)^2$. Using this result with $x = (1-p)$ shows that $E(X) = 1/p$, as expected.

10. Let $X_n$ be the number of times that a player’s money doubles in $n$ tries. By construction $X_n \sim Bin(n, 1/2)$ and $Pr(X_n = x) = \binom{n}{x}(1/2)^n$. If our money doubles $x$ times, it necessarily halved $n - x$ times. As a result, if we win $x$ times then our fortune is $C2^x/2^{n-x} = C2^{2x-n}$. Combined with the definition of expectation, this means that the expected fortune after $n$ tries is

\[
\sum_{x=0}^{n} C2^{2x-n} \binom{n}{x} (1/2)^n = C/2^n \sum_{x=0}^{n} \binom{n}{x} 2^x (1/2)^{n-x} \\
= C/2^n (2 + 1/2)^n = C(5/4)^n.
\]

If I asked to play the game with you where you pay me if the fortune increases, and I pay you if it decreases, would you do it?

11. Let $X_n$ be the total number of times that the particle jumps to the right in $n$ tries. By construction $X_n \sim Bin(n, p)$ and $Pr(X_n = x) = \binom{n}{x}p^x(1 -$
$p)^{n-x}$. If the particle jumps right $x$ times, then it jumps to the left $n - x$ times and the position is $x - (n - x) = 2x - n$. By the definition of expectation, the expected position after $n$ tries is $\sum_{x=0}^{n} (2x - n) Pr(X_n = x) = n(2p - 1)$. The second moment can be calculated from $\sum_{x=0}^{n} (2x - n)^2 Pr(X_n = x) = n(2p - 1)$. Combining the two gives the variance of the position $= 4np(1 - p)$. 