Midterm Exam Solutions

- The exam is closed book.
- The exam will end promptly at 12:30. There will be no exceptions.
- Each problem is worth 10 points, and partial credit is possible.
- Below are some potentially useful pdfs and facts. You may use these without proof.

1. Consider the pdf, $f(x; \lambda) = \frac{1}{\lambda} \exp(-x/\lambda), x \geq 0$  $E(X) = \lambda, Var(X) = \lambda^2$

2. $InverseGamma(\gamma, \delta)$: $f(x; \gamma, \delta) = \frac{\delta^\gamma x^{-\gamma-1} \exp(-\delta/x)}{\Gamma(\gamma)}, x > 0, \delta > 0, \gamma > 0.$  
   $E(X) = \delta/(\gamma - 1), \gamma > 1$

3. $k$th order stat from $n$ iid obs from density $g(x)$: $f(x) = \frac{n!g(x)G(x)^{k-1}(1-G(x))^{n-k}}{(k-1)!(n-k)!}$

If $Z \sim Exponential(\lambda)$, then $Pr(Z > z) = \exp(-z/\lambda)$. An $z$ so that $Pr(Z > z) = \alpha$ is $z = -\lambda \log(\alpha)$. (This has been corrected from what was on the exam!)

1. Suppose $x_1, \ldots, x_n$ is an iid sample from the $Exponential(\beta)$ distribution.
   - (a) Show that $\hat{\beta}_{MLE} = \bar{X}$
     The log likelihood is $l(\beta, x) = -n \log \beta - \sum x_i/\beta$, and the score equation is $0 = -n/\beta + \sum x_i/\beta^2$ which is solved by $\hat{\beta} = \bar{x}$. Since the second derivative of the log likelihood is $n/\beta^2 - 2\sum x_i/\beta^3$ which has value $-n$ at $\hat{\beta}$ and the likelihood goes to $-\infty$ as $\beta$ goes to zero or $\infty$, $\hat{\beta}$ is the MLE.
   - (b) Show that the Cramer-Rao lower bound for an unbiased estimator of $\beta$ is $\beta^2/n$.
     The CRLB is $\frac{1}{Var(l''(\beta, x_i))}$. Using the previous answer, $E(l''(\beta, x_i)) = E(1/\beta^2 - 2x_i/\beta^3) = 1/\beta^2 - 2\beta/\beta^3 = -1/\beta^2$. The answer follows.
   - (c) Propose an estimator for $\beta$ whose variance is less than the Cramer-Rao lower bound.
     This was an unfortunately worded question. All get credit for it. (It should have said that "less than the Cramer-Rao lower bound for an unbiased estimator.")
   - (d) Suppose the prior on $\beta$ is $InverseGamma(\gamma, \delta)$, and find the Bayes estimator of $\beta$.
     The Bayes estimator is the posterior mean. The posterior distribution is $f(\beta|x) = f(x, \beta)\pi(\beta)/C$ where $\pi(\beta)$ is the prior and $C$ is a constant that doesn’t depend on $\beta$. $f(x, \beta)\pi(\beta) = \beta^{-n} \exp(-\sum x_i/\beta)\frac{\delta^{\gamma-1} \exp(-\delta/x)}{\Gamma(\gamma)}$. As a result, $f(\beta|x) = \beta^{-n-1} \exp(-\sum x_i + \delta)/\beta)/\Gamma(\gamma) + \delta)/\gamma + n - 1.$ This is a kernel of an $InverseGamma(\gamma + n, \sum x_i + \delta)$ so the posterior mean is $(\sum x_i + \delta)/(\gamma + n - 1)$.

2. Consider the pdf, $f(x; \alpha) = \exp\{-\alpha \cdot x\}1_{x>\alpha}$, and suppose $x_1, \ldots, x_n$ is an iid sample from that distribution. Hint: For (b), (c), and (d), you may use without proof that if $Y = X - \alpha$ then $Y \sim Exponential(1)$.
(a) Find a maximum likelihood estimator for $\alpha$.

The likelihood is $L(\alpha; x) = \exp \left\{ -\left( \sum x_i - n\alpha \right) \right\} 1_{\min(x) > \alpha}$. This is an increasing function for any $\alpha$ less than or equal to $\min(x)$, and it is zero for $\alpha > \min(x)$. As a result, the MLE is $x^{(1)} = \min(x)$.

(b) Modify the MLE to get an unbiased estimator of $\alpha$.

Application of the order statistic density with $k = 1$ shows that $X^{(1)} - \alpha \sim \text{Exponential}(1/n)$. As a result, $E(\hat{\alpha}_{MLE}) = E(X^{(1)}) = \alpha + 1/n$, and $\hat{\alpha}_{MLE} - 1/n$ is unbiased.

(c) Show that a MOM estimator of $\alpha$ is $X - 1$.

Let $Y = X - \alpha$, then $Y \sim \text{Exponential}(1)$ and $E(Y) = E(X) - \alpha$. Since $E(Y) = 1$, $E(X) = 1 + \alpha$, and $\hat{\alpha}_{MOM} = \overline{x} - 1$.

(d) Find the MSE of $\hat{\alpha}_{MOM}$.

By construction $E(\hat{\alpha}_{MOM}) = 1 + \alpha - 1 = \alpha$. Since $\text{Var}(X) = \text{Var}(Y + \alpha) = \text{Var}(Y) = 1$, $\text{Var}(\hat{\alpha}_{MOM}) = \text{Var}(\overline{x} - 1) = \text{Var}(\overline{x}) = 1/n$. The MSE is $(E(\hat{\alpha}_{MOM} - \alpha))^2 + \text{Var}(\hat{\alpha}_{MOM}) = 0 + 1/n$.

(e) Suppose you want to use a likelihood ratio test to test $H_0 : \alpha = 0$ vs $H_A : \alpha > 0$.

i. Show that the likelihood ratio test rejects when $X^{(1)}$ is sufficiently large.

$$LR(x) = \frac{\max_{\alpha \geq 0} L(\alpha; x)}{\max_{\alpha > 0} L(\alpha; x)}.$$ If $\hat{\alpha}_{MLE} \leq 0$, then $LR(x) = 1$. If $\hat{\alpha}_{MLE} > 0$, then $LR(x) = \exp(-\sum x_i) / \exp(-\sum x_i + nx^{(1)}) = \exp(-nx^{(1)})x^{(1)} > 0$, which is smaller than some cutoff if $x^{(1)}$ is sufficiently large.

ii. What is the cutoff for a level 0.05 likelihood ratio test?

For a level $\alpha$ test, we need $c$ such that $P_r(LR(X) < c) = \alpha$ when $H_0$ is true. Since $P_r(LR(X) < c) = P_r(\exp(-nX^{(1)}) < c) = P_r(X^{(1)} > -\log(c)/n)$ the problem becomes: find $C$ so that $P_r(X^{(1)} > C) = \alpha$. When $\alpha = 0$, $X^{(1)} \sim \text{Exponential}(1/n)$ and $P_r(X^{(1)} > C) = \exp(-nC)$ and $C = -\log(\alpha)/n$. 

\begin{align*}
\text{(a) } &\text{Find a maximum likelihood estimator for } \alpha. \\
&\text{The likelihood is } L(\alpha; x) = \exp \{-\left( \sum x_i - n\alpha \right) \} 1_{\min(x) > \alpha}. \text{ This is an increasing function for any } \alpha \text{ less than or equal to } \min(x), \text{ and it is zero for } \alpha > \min(x). \text{ As a result, the MLE is } x^{(1)} = \min(x). \\
\text{(b) } &\text{Modify the MLE to get an unbiased estimator of } \alpha. \\
&\text{Application of the order statistic density with } k = 1 \text{ shows that } X^{(1)} - \alpha \sim \text{Exponential}(1/n). \text{ As a result, } E(\hat{\alpha}_{MLE}) = E(X^{(1)}) = \alpha + 1/n, \text{ and } \hat{\alpha}_{MLE} - 1/n \text{ is unbiased.} \\
\text{(c) } &\text{Show that a MOM estimator of } \alpha \text{ is } X - 1. \\
&\text{Let } Y = X - \alpha, \text{ then } Y \sim \text{Exponential}(1) \text{ and } E(Y) = E(X) - \alpha. \text{ Since } E(Y) = 1, E(X) = 1 + \alpha, \text{ and } \hat{\alpha}_{MOM} = \overline{x} - 1. \\
\text{(d) } &\text{Find the MSE of } \hat{\alpha}_{MOM}. \\
&\text{By construction } E(\hat{\alpha}_{MOM}) = 1 + \alpha - 1 = \alpha. \text{ Since } \text{Var}(X) = \text{Var}(Y + \alpha) = \text{Var}(Y) = 1, \text{ Var}(\hat{\alpha}_{MOM}) = \text{Var}(\overline{x} - 1) = \text{Var}(\overline{x}) = 1/n. \text{ The MSE is } (E(\hat{\alpha}_{MOM} - \alpha))^2 + \text{Var}(\hat{\alpha}_{MOM}) = 0 + 1/n. \\
\text{(e) } &\text{Suppose you want to use a likelihood ratio test to test } H_0 : \alpha = 0 \text{ vs } H_A : \alpha > 0. \\
&\text{i. Show that the likelihood ratio test rejects when } X^{(1)} \text{ is sufficiently large.} \\
&LR(x) = \frac{\max_{\alpha \geq 0} L(\alpha; x)}{\max_{\alpha > 0} L(\alpha; x)}. \text{ If } \hat{\alpha}_{MLE} \leq 0, \text{ then } LR(x) = 1. \text{ If } \hat{\alpha}_{MLE} > 0, \text{ then } LR(x) = \exp(-\sum x_i) / \exp(-\sum x_i + nx^{(1)}) = \exp(-nx^{(1)})x^{(1)} > 0, \text{ which is smaller than some cutoff if } x^{(1)} \text{ is sufficiently large.} \\
&\text{ii. What is the cutoff for a level 0.05 likelihood ratio test?} \\
&\text{For a level } \alpha \text{ test, we need } c \text{ such that } P_r(LR(X) < c) = \alpha \text{ when } H_0 \text{ is true. Since } P_r(LR(X) < c) = P_r(\exp(-nX^{(1)}) < c) = P_r(X^{(1)} > -\log(c)/n) \text{ the problem becomes: find } C \text{ so that } P_r(X^{(1)} > C) = \alpha. \text{ When } \alpha = 0, \text{ } X^{(1)} \sim \text{Exponential}(1/n) \text{ and } P_r(X^{(1)} > C) = \exp(-nC) \text{ and } C = -\log(\alpha)/n.} \end{align*}