MATH 552 **Scientific Computing II** **Spring 2016**

Homework Set 3 - SOLUTIONS

1. *Spectrum of Skew-Hermitian Matrices*: An $n \times n$ complex matrix A is said to be *skew-Hermitian* if $A^* = \bar{A}^T = -A$. If A is real, this reduces to $A^T = -A$.

   Show that the eigenvalues of a skew-Hermitian matrix are pure imaginary, i.e. $\bar{\lambda} = -\lambda$.

   <u>ANS</u>: Suppose $Ax = \lambda x$ for $x \neq 0$, i.e. (λ, x) is an eigenpair of A. Then $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle$, or $\lambda = \langle Ax, x\rangle / \langle x, x\rangle$, and

$$\overline{\lambda} = \frac{\overline{\langle Ax, x\rangle}}{\overline{\langle x, x\rangle}} = \frac{\langle x, Ax\rangle}{\langle x, x\rangle} = \frac{\langle A^* x, x\rangle}{\langle x, x\rangle} = \frac{\langle -Ax, x\rangle}{\langle x, x\rangle} = -\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = -\lambda.$$

2. Consider the discrete eigenproblem for $-D^2$, the $O(h^2)$ approximation to $-d/dx^2$. To this end, choose $N > 0$, let $h = 1/N$ and $x_i = i * h$ for $i = 0, 1, \ldots, N$. Note we now have $N+1$ grid points with $x_0 = 0$ and $x_N = 1$. So we seek e-pairs which satisfy

$$(-D^2 v)_i = \frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} = \lambda v_i \quad \text{for} \quad i = 1, 2, \ldots, N-1,$$

with $v_0 = v_N = 0$. In matrix form,

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}$$

Show that (λ_k, v_k) is an e-pair for $k = 1, 2, \ldots, N-1$ where $\lambda_k = 2(1 - \cos k\pi h)/h^2$ and v_k is the vector with components $(v_k)_i = \sin ik\pi h$. Hint: Use trig identities, and note that $\sin 0k\pi h = \sin Nk\pi h = 0$.

.**<u>ANS</u>**:

Noting that $\sin 0k\pi h = \sin Nk\pi h = 0$. Fix a k, $k = 1, \ldots, N-1$. For $i = 1, \ldots, N-1$,

$$\begin{aligned}
\frac{-v_{k,i-1} + 2v_{k,i} - v_{k,i+1}}{h^2} &= \frac{-\sin(i-1)k\pi h + 2\sin ik\pi h - \sin(i+1)k\pi h}{h^2} \\
&= \frac{-(\sin ik\pi h \cos k\pi h - \cos ik\pi h \sin k\pi h) + 2\sin ik\pi h - (\sin ik\pi h \cos k\pi h + \cos ik\pi h \sin k\pi h)}{h^2} \\
&= \frac{-2\sin ik\pi h \cos k\pi h + 2\sin ik\pi h}{h^2} \\
&= \frac{2(1-\cos k\pi h)}{h^2}\sin ik\pi h \\
&= \lambda_k \sin ik\pi h. \\
&= \lambda_k v_{k,i}.
\end{aligned}$$

So we have $Av_k = \lambda_k v_k$ for $k = 1, \ldots, N-1$, where A is the matrix representing $-D^2$. Thus, (λ_k, v_k) are e-pairs for $k = 1, \ldots, N-1$.

3. Given a real vector $v = (v_1, \ldots, v_{N-1})^T$ the *Discrete Sine Transform* of v is given by $\hat{v} = P^{-1}v$, where P is an $(N-1) \times (N-1)$ matrix with $P_{i,j} = (2/\sqrt{2N})\sin(ij\pi/N)$ for $i, j = 1, 2, \ldots, N-1$. Show $P = P^T$ and $P^{-1} = P$.

<u>**ANS**</u>: **(a)** First, $P_{i,j} = (2/\sqrt{2N})\sin(ij\pi/N) = (2/\sqrt{2N})\sin(ji\pi/N) = P_{j,i} \quad \Rightarrow \quad P = P^T$.

To show $P^{-1} = P$ we compute P^2. Suppose $l \neq j$, then

$$\begin{aligned}
(P^2)_{l,j} &= \sum_{k=1}^{N-1} \frac{2}{\sqrt{2N}} \sin(lk\pi/N) \frac{2}{\sqrt{2N}} \sin(kj\pi/N) \\
&= \frac{1}{N} \sum_{k=1}^{N-1} 2\sin(lk\pi/N)\sin(kj\pi/N) \\
&= \frac{1}{N} \sum_{k=1}^{N-1} [\cos(k(l-j)\pi/N) - \cos(k(l+j)\pi/N)] \\
&= \frac{1}{2N}\left[\sum_{k=1}^{N-1} 2\cos(k(l-j)\pi/N) - \sum_{k=1}^{N-1} 2\cos(k(l+j)\pi/N)\right] \\
&= \frac{1}{2N}\left[\sum_{k=1}^{N-1} (e^{i(k(l-j)\pi/N)} + e^{-i(k(l-j)\pi/N)}) - \sum_{k=1}^{N-1} (e^{i(k(l+j)\pi/N)} + e^{-i(k(l+j)\pi/N)})\right] \\
&= \frac{1}{2N}\left[\sum_{k=1}^{N-1} [(e^{\frac{i(l-j)\pi}{N}})^k + (e^{\frac{-i(l-j)\pi}{N}})^k] - \sum_{k=1}^{N-1} [(e^{\frac{i(l+j)\pi}{N}})^k + (e^{\frac{-i(l+j)\pi}{N}})^k]\right] \\
&= \frac{1}{2N}\left[\frac{1-(e^{\frac{i(l-j)\pi}{N}})^N}{1-e^{\frac{i(l-j)\pi}{N}}} - 1 + \frac{1-(e^{\frac{-i(l-j)\pi}{N}})^N}{1-e^{\frac{-i(l-j)\pi}{N}}} - 1\right] - \\
&\quad \frac{1}{2N}\left[\frac{1-(e^{\frac{i(l+j)\pi}{N}})^N}{1-e^{\frac{i(l+j)\pi}{N}}} - 1 + \frac{1-(e^{\frac{-i(l+j)\pi}{N}})^N}{1-e^{\frac{-i(l+j)\pi}{N}}} - 1\right] \\
&= \frac{1}{2N}\left[\left(\frac{1-e^{i(l-j)\pi}}{1-e^{i\frac{(l-j)\pi}{N}}} + \frac{1-e^{-i(l-j)\pi}}{1-e^{-i\frac{(l-j)\pi}{N}}}\right) - \left(\frac{1-e^{i(l+j)\pi}}{1-e^{i\frac{(l+j)\pi}{N}}} + \frac{1-e^{-i(l+j)\pi}}{1-e^{-i\frac{(l+j)\pi}{N}}}\right)\right] \\
&= \frac{1}{2N}\left[\left(\frac{1-\cos(l-j)\pi}{1-e^{i\frac{(l-j)\pi}{N}}} + \frac{1-\cos(l-j)\pi}{1-e^{-i\frac{(l-j)\pi}{N}}}\right) - \left(\frac{1-e^{i(l+j)\pi}}{1-e^{i\frac{(l+j)\pi}{N}}} + \frac{1-e^{-i(l+j)\pi}}{1-e^{-i\frac{(l+j)\pi}{N}}}\right)\right].
\end{aligned}$$

Letting $\theta = (l-j)\pi$, the first term in the brackets above is

$$\frac{1-\cos\theta}{1-e^{i\frac{\theta}{N}}} + \frac{1-\cos\theta}{1-e^{-i\frac{\theta}{N}}} = (1-\cos\theta)\frac{1-e^{i\frac{\theta}{N}} + 1 - e^{-i\frac{\theta}{N}}}{(1-e^{i\frac{\theta}{N}})(1-e^{-i\frac{\theta}{N}})} = (1-\cos\theta)\,.$$

A similar calculation shows that the second term is $(1-\cos(l+j)\pi)$. Putting everything together, for $l \neq j$,

$$(P^2)_{l,j} = \frac{1}{2N}\left[(1-\cos(l-j)\pi) - (1-\cos(l+j)\pi)\right] = \frac{1}{2N}\left[\cos(l+j)\pi) - \cos(l-j)\pi)\right]\,.$$

Since l, j are integers, either $l-j$ and $l+j$ are **both** even or they are **both** odd (if $l-j$ is even then $l+j = l-j+(2j)$ is even, etc.). Thus (finally!), $(P^2)_{l,j} = 0$ for $l \neq j$.

If $l = j$ the full derivation above is not valid since once we summed the partial series the denominators in the first term are zero. However, since $1 \leq l, j \leq N-1$ we must have that $l + j < 2N$, thus the denominators in the second term are never zero. So going back a few steps in the derivation above,

$$
\begin{aligned}
(P^2)_{l,j} &= \frac{1}{2N}\left[\sum_{k=1}^{N-1} 2\cos\left(k(l-j)\pi/N\right) - \sum_{k=1}^{N-1} 2\cos\left(k(l+j)\pi/N\right)\right] \\
&= \frac{1}{2N}\left[\sum_{k=1}^{N-1} 2\cos\left(k0\pi/N\right) - \sum_{k=1}^{N-1} 2\cos\left(k(l+j)\pi/N\right)\right] \\
&= \frac{1}{2N}\left[2(N-1) - (1 - \cos\left((l+j)\pi\right) - 2)\right] \\
&= \frac{1}{2N}\left[2(N-1) - (1 - \cos\left(2l\pi\right) - 2)\right] = \frac{1}{2N}\left[2(N-1) - (1 - 1 - 2)\right] = 1
\end{aligned}
$$

Thus, $P^2 = I \quad \Rightarrow \quad P^{-1} = P$. There are most likely more concise ways to show this result.

4. *Boundary Value Problems and Boundary Layers*: Consider the two-point boundary value problem

$$\begin{cases} -\epsilon u'' + u = 2x + 1, 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

where $\epsilon > 0$ is a given parameter. The exact solution is given by

$$u(x) = 2x + 1 - \frac{\sinh \frac{1-x}{\sqrt{\epsilon}} + 3 \sinh \frac{x}{\sqrt{\epsilon}}}{\sinh \frac{1}{\sqrt{\epsilon}}}$$

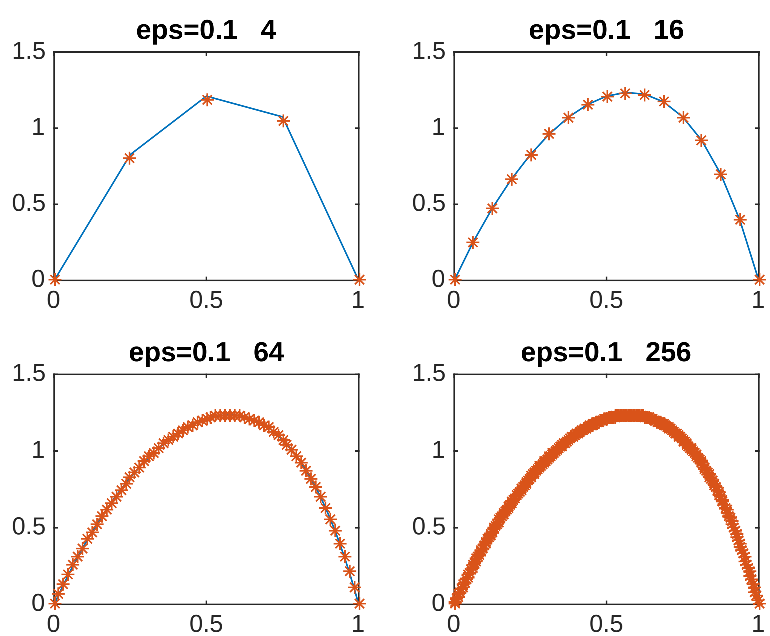
(a) Using your tridiagonal solver compute the solution for $\epsilon = 10^{-1}$ and $N = 1/h = 4^n$ for $n = 1, 2, 3, 4$. Using the *subplot* command, plot the exact solution and the computed solution for each N on the same page, i.e. 4 plots on the same page. Also, compute the ratios $\|u - v\|_\infty / h^2$ for each $h = 1/N$. Discuss the results. Include a copy of your code.

(b) Repeat the exercise above for $\epsilon = 10^{-3}$. Again, discuss the results. What has changed, i.e. what is the effect of a smaller ϵ?

ANS: Here is the output for $\epsilon = 0.1$:

h	inf_error	inf_error/h^2
2.5000e-01	2.8851e-02	4.6161e-01
6.2500e-02	1.9913e-03	5.0977e-01
1.5625e-02	1.2518e-04	5.1273e-01
3.9062e-03	7.8260e-06	5.1288e-01

along with the graph

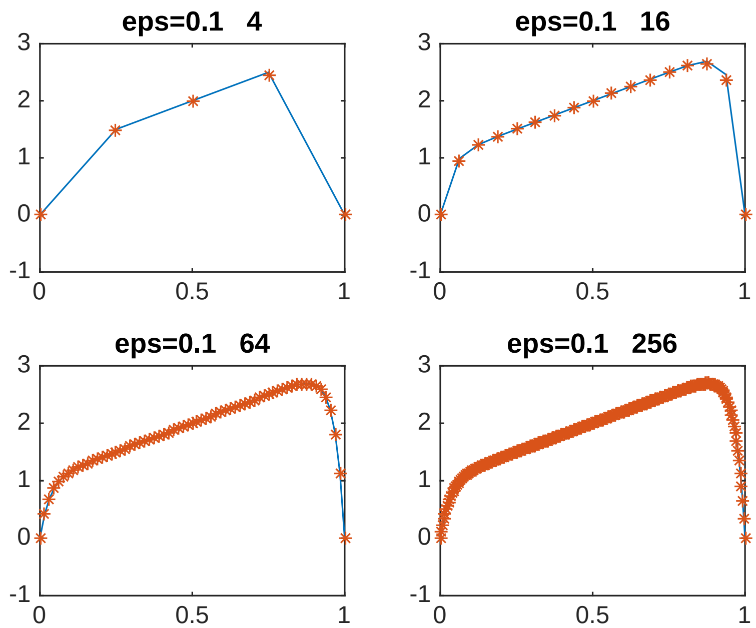


We can see from the last column in the table that the numbers are converging, so we are achieving $O(h^2)$ accuracy.

Now for $\epsilon = 0.001$ the results are:

```
       h          inf_error    inf_error/h^2
-----------------------------------------
   2.5000e-01    4.5421e-02    7.2673e-01
   6.2500e-02    1.0771e-01    2.7573e+01
   1.5625e-02    1.0982e-02    4.4983e+01
   3.9062e-03    7.0064e-04    4.5917e+01
```

and the graph



We can see in the graphs that the solution is now quite steep at each bounderies. Specifically, since we are using equi-spaced points, the scheme has some trouble resolving these regions. For example, for $N = 16$ we see there are only 2 points in the boundary layer region.

This is also indicated in the table. While the last column appears to be converging to a constant, the convergence is not as clear as was for the $\epsilon = 0.1$ case above.

Here is a copy of the code:

```
n = [1:4]';
dispvars = zeros(length(n),3);

epsilon = 10^(-3);
sqrteps = sqrt(epsilon);
denom = sinh(1/sqrteps);
```

```
figure(1)

for i = 1:length(n)
    N = 4^(n(i));
    h = 1/N;
    xh = h*(0:N)';

    u_h = 2*xh+1 - ...
          (sinh((1-xh)/sqrteps)+3*sinh(xh/sqrteps))/denom;  % true soln u
    f_h = 2*xh+1;                          % -epsilon*u'' + u = f

    a = epsilon*(2*ones(N-1,1)/(h^2))+1;   % create a, b & c
    b = epsilon*(-ones(N-1,1)/(h^2));
    c = epsilon*(-ones(N-1,1)/(h^2));
    ftil = f_h(2:N);         % rhs is f evaluated at n-1 interior pts

    v_h = trisolve(a,b,c,ftil);  %  solve
    v_h = [0;v_h;0];             %  set BCs u(0)=u(1)=0

    dispvars(i,1) = h;
    dispvars(i,2) = max(abs(v_h-u_h));
    dispvars(i,3) = dispvars(i,2)/h^2;

    subplot(2,2,i)
    plot(xh,u_h,'-',xh,v_h,'*')
    title(['eps=0.1   ' num2str(N)]);
end

save2pdf('hw3_p4b.pdf',1,300)

 format short e
 disp(' ')
 disp('      h        inf_error   inf_error/h^2')
 disp('----------------------------------------')
 disp(dispvars)
```