MATH 552 **Scientific Computing II** **Spring 2016**

Homework Set 3 Due Friday, 4 March 2016

1. *Spectrum of Skew-Hermitian Matrices*: An $n \times n$ complex matrix A is said to be *skew-Hermitian* if $A^* = \bar{A}^T = -A$. If A is real, this reduces to $A^T = -A$.

   Show that the eigenvalues of a skew-Hermitian matrix are pure imaginary, i.e. $\bar{\lambda} = -\lambda$.

2. Consider the discrete eigenproblem for $-D^2$, the $O(h^2)$ approximation to $-d/dx^2$. To this end, choose $N > 0$, let $h = 1/N$ and $x_i = i * h$ for $i = 0, 1, \ldots, N$. Note we now have $N+1$ grid points with $x_0 = 0$ and $x_N = 1$. So we seek e-pairs which satisfy

$$(-D^2 v)_i = \frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} = \lambda v_i \quad \text{for} \quad i = 1, 2, \ldots, N-1,$$

   with $v_0 = v_N = 0$. In matrix form,

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}$$

   Show that (λ_k, v_k) is an e-pair for $k = 1, 2, \ldots, N-1$ where $\lambda_k = 2(1 - \cos k\pi h)/h^2$ and v_k is the vector with components $(v_k)_i = \sin ik\pi h$. Hint: Use trig identities, and note that $\sin 0k\pi h = \sin Nk\pi h = 0$.

   .

3. Given a real vector $v = (v_1, \ldots, v_{N-1})^T$ the *Discrete Sine Transform* of v is given by $\hat{v} = P^{-1}v$, where P is an $(N-1) \times (N-1)$ matrix with $P_{i,j} = (2/\sqrt{2N}) \sin(ij\pi/N)$ for $i, j = 1, 2, \ldots, N-1$. Show $P = P^T$ and $P^{-1} = P$.

4. *Boundary Value Problems and Boundary Layers*: Consider the two-point boundary value problem

$$\begin{cases} -\epsilon u'' + u = 2x + 1, 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

   where $\epsilon > 0$ is a given parameter. The exact solution is given by

$$u(x) = 2x + 1 - \frac{\sinh \frac{1-x}{\sqrt{\epsilon}} + 3 \sinh \frac{x}{\sqrt{\epsilon}}}{\sinh \frac{1}{\sqrt{\epsilon}}}$$

   (a) Using your tridiagonal solver compute the solution for $\epsilon = 10^{-1}$ and $N = 1/h = 4^n$ for $n = 1, 2, 3, 4$. Using the *subplot* command, plot the exact solution and the computed solution for each N on the same page, i.e. 4 plots on the same page. Also, compute the ratios $\|u - v\|_\infty / h^2$ for each $h = 1/N$. Discuss the results. Include a copy of your code.

   (b) Repeat the exercise above for $\epsilon = 10^{-3}$. Again, discuss the results. What has changed, i.e. what is the effect of a smaller ϵ?