MATH 552 **Scientific Computation II** **Spring 2016**

Homework Set 2 - SOLUTIONS

1. *Orthogonality of complex exponentials*: Consider the complex exponential functions

$$\phi_n(x) = e^{-i(\frac{n\pi x}{L})} \quad \text{for} \quad -\infty < n < \infty\,.$$

Show that

$$\langle \phi_n(x), \phi_m(x)\rangle = \int_{-L}^{L} \phi_n(x)\overline{\phi_m}(x)\ dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } n = m\,, \end{cases}$$

and thus the functions are mutually orthogonal.

ANS: First note that

$$\phi_n(x) = e^{-i(\frac{n\pi x}{L})} = \cos\left(-\frac{n\pi x}{L}\right) + i\sin\left(-\frac{n\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right).$$

Thus,

$$\begin{aligned}
\langle \phi_n(x), \phi_m(x)\rangle &= \int_{-L}^{L} \phi_n(x)\overline{\phi_m}(x)\ dx \\
&= \int_{-L}^{L} \left(\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right)\left(\cos\left(\frac{m\pi x}{L}\right) + i\sin\left(\frac{m\pi x}{L}\right)\right)\ dx \\
&= \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx \\
&\quad + i\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) - \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx \\
&= \begin{cases} (0+0) + i(0-0) & n \neq m \\ (L+L) + i(0-0) & n = m \end{cases} \\
&= \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}
\end{aligned}$$

due to the mutual orthogonality of Sines and Cosines over $[-L, L]$. This was shown in HW #1 when $L = \pi$. However, the argument in that solution follows for any $L > 0$.

2. Show that the second order centered finite difference approximation to $u''(x_j)$,

$$D^2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2},$$

satisifes

$$u''(x_j) = D^2 u_j + O(h^2),$$

using Taylor series approximations. Here $u_j = u(x_j)$ and $u_{j\pm1} = u(x_{j\pm1})$ where $x_{j\pm1} = x_j \pm h$. Derive a concise formula for the $O(h^2)$ error term.

<u>ANS</u>: We have the following Taylor expansions:

$$\begin{aligned}
u_{j-1} &= u(x_j) - u'(x_j)h + \tfrac{1}{2}u''(x_j)h^2 - \tfrac{1}{6}u'''(x_j)h^3 + \tfrac{1}{24}u^{(4)}(c_x^-)h^3 \quad \text{where } c_x^- \in (x_j - h, x_j) \\
u_j &= u(x_j) \\
u_{j+1} &= u(x_j) + u'(x_j)h + \tfrac{1}{2}u''(x_j)h^2 + \tfrac{1}{6}u'''(x_j)h^3 + \tfrac{1}{24}u^{(4)}(c_x^+)h^3 \quad \text{where } c_x^+ \in (x_j, x_j + h)
\end{aligned}$$

Forming the linear combination gives: $au_{j-1} + bu_j + cu_{j+1} =$

$$(a + b + c)u_j + (-ah + ch)u'(x_j) + \frac{1}{2}(ah^2 + ch^2)u''(x_j) + \ldots$$

Since we have three unknowns a, b, and c, we choose them so that $u''(x_j)$ is multiplied by 1, and $u(x_j)$ and $u'(x_j)$ are multiplied by 0. Thus a, b, and c must satisfy

$$\begin{aligned}
a + b + c &= 0 \\
-ah + ch &= 0 \quad \Rightarrow \quad a = c = \tfrac{1}{h^2},\ b = -\tfrac{2}{h^2}. \\
(ah^2 + ch^2)/2 &= 1
\end{aligned}$$

Using these values gives us the approximation

$$u''(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}.$$

Since $a = c$ the coefficient for the $u'''(x_j)$ term is 0, and the error term is then given by

$$\frac{1}{24}h^4(au^{(4)}(c_x^-) + cu^{(4)}(c_x^+)) = \frac{1}{24}h^2(u^{(4)}(c_x^-) + u^{(4)}(c_x^+))$$

and assuming that $u^{(4)}(x)$ is a continuous then $u^{(4)}(c_x^-) + u^{(4)}(c_x^+) = 2u^{(4)}(c_x^*)$ where $c_x^* \in (c_x^-, c_x^+)$ function. Then the error term can be written

$$\frac{1}{24}h^2(2u^{(4)}(c_x^*)) = \frac{h^2}{12}u^{(4)}(c_x^*) = O(h^2).$$

3. Find **by hand** the eigenvalues and eigenvectors of the following matrices
(a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

<u>**ANS:**</u> We first determine the eigenvalues from the roots of $p_A(\lambda) = det(A - \lambda I)$, the characteristic polynomial, and then for each a corresponding non-zero eigenvector x which is a solution of the homogeneous system $(A - \lambda I)x = 0$.

(a) $p_A(\lambda) = det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 1.$
For $\lambda_1 = 2$ we have

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 1-2 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x = 0 \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_1}, \mathbf{x_1}) = (\mathbf{2}, [\mathbf{1} \;\; \mathbf{0}]^{\mathbf{T}})$. For $\lambda_1 = 1$ we have

$$\begin{bmatrix} 2-1 & 0 \\ 0 & 1-1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} 0 \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_2}, \mathbf{x_2}) = (\mathbf{1}, [\mathbf{0} \;\; \mathbf{1}]^{\mathbf{T}})$. Of course, for this matrix we can easily see that these are the eigenvalues and eigenvectors since it is diagonal.

(b) $p_A(\lambda) = det(A - \lambda I) = det \begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 2 = 0 \Rightarrow \lambda_1 = \sqrt{2}, \lambda_2 = -\sqrt{2}$. For $\lambda_1 = \sqrt{2}$ we have

$$\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} x = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} x = 0 \Rightarrow x^{(1)} = \sqrt{2}x^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} c\sqrt{2} \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_1}, \mathbf{x_1}) = (\sqrt{\mathbf{2}}, [\sqrt{\mathbf{2}} \;\; \mathbf{1}]^{\mathbf{T}})$. For $\lambda_2 = -\sqrt{2}$ we have

$$\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} x = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} x = 0 \Rightarrow x^{(1)} = -\sqrt{2}x^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} -c\sqrt{2} \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_2}, \mathbf{x_2}) = (-\sqrt{\mathbf{2}}, [-\sqrt{\mathbf{2}} \;\; \mathbf{1}]^{\mathbf{T}})$.

(c) $p_A(\lambda) = det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$. For $\lambda_1 = 1$ we have

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x = 0 \Rightarrow x^{(1)} = x^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} c \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_1}, \mathbf{x_1}) = (\mathbf{1}, [\mathbf{1} \;\; \mathbf{1}]^{\mathbf{T}})$. For $\lambda_2 = 3$ we have

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x = 0 \Rightarrow x^{(1)} = -x^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} -c \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_2}, \mathbf{x_2}) = (\mathbf{3}, [-\mathbf{1} \;\; \mathbf{1}]^{\mathbf{T}})$.

(d) $p_A(\lambda) = det(A - \lambda I) = det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \lambda_2 = -i$. For $\lambda_1 = i$ we have

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} x = 0 \Rightarrow x^{(1)} = ix^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} ic \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_1}, \mathbf{x_1}) = (\mathbf{i}, [\mathbf{i} \ \ \mathbf{1}]^{\mathbf{T}})$. For $\lambda_2 = -i$ we have

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} x = 0 \Rightarrow x^{(1)} = -ix^{(2)} \Rightarrow x \in \text{ span } \left\{ \begin{bmatrix} -ic \\ c \end{bmatrix} \right\} \text{ for any } c \neq 0.$$

Taking $c = 1$, we have the eigen-pair $(\boldsymbol{\lambda_2}, \mathbf{x_2}) = (-\mathbf{i}, [-\mathbf{i} \ \ \mathbf{1}]^{\mathbf{T}})$.

4. Write a MATLAB function M-file **trisolve.m** to solve the linear system $Ax = f$ where

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_n & a_n \end{pmatrix}$$

is a tridiagonal $n \times n$ matrix **Assume** that no partial pivoting is required. The inputs are the n-vectors a, b, c and f and returns the solution x. Its first line should read:

```
function x = trisolve(a,b,c,f)
```

Test your code with the 5×5 system with $a_i = 2$, $b_i = -1$, $c_i = -1$, and RHS $f = [1, 0, 0, 0, 1]^T$. The exact solution is $x = [1, 1, 1, 1, 1]^T$. Use MATLAB's **diary** command to save your MATLAB session output showing that your code works properly. Include a copy of both codes.

ANS: Here is the MATLAB code

```
function x = trisolve(a,b,c,f)
%
%  A = tridiag(b,a,c)

n = length(a);                  % determine system size
x = zeros(n,1);                 % allocate x

for i = 1:n-1                   % forward eliminatiosweep GE
    m = b(i+1)/a(i);            % compute multiplier
    a(i+1) = a(i+1)-m*c(i);     % a (but not c) changes for row operation
    f(i+1) = f(i+1)-m*f(i);     % row op applied to RHS f
end

x(n) = f(n)/a(n);               % back substitution
for i = n-1:-1:1
    x(i) = (f(i)-c(i+1)*x(i+1))/a(i);
end
```

and here is the output of the $5x5$ test system:

```
>> a = 2*ones(5,1); b = -ones(5,1); c=b; f=[1 0 0 0 1]';
>> x = trisolve(a,b,c,f)

x =

    1.0000
    1.0000
    1.0000
    1.0000
    1.0000
```

5. Consider the 2-point BVP

$$\begin{cases} -u'' = -(x^2 + 3x)e^x \\ u(0) = u(1) = 0 \end{cases}$$

(a) Show $u(x) = (x^2 - x)e^x$ is the exact solution.

(b) Write a MATLAB function M-file to solve the problem using the 2nd order centered FD scheme we discussed in class, $-D^2 v_i = f_i$, that utilizes your m-file **trisolve.m** from problem 3 above. Note that $\sigma = 0$ here. Assume a mesh size $h = 1/n$ where $n = 2^p$ for p a positive integer. For $p = 1 : 12$ present a table with the following data - column 1: h; column 2: $\|u_h - v_h\|_\infty$; column 3: $\|u_h - v_h\|_\infty / h^2$; where $h = 1/n$. What does the trend in the third column indicate? Include a copy of your code.

<u>ANS</u>: (a) is just straightforward substitution.

Here is the output and code for (b). Note that the third column is approaching a constant, indicating second order convergence.

```
        h          inf_error    inf_error/h^2
-----------------------------------------
   5.0000e-01   5.1523e-02   2.0609e-01
   2.5000e-01   1.3185e-02   2.1096e-01
   1.2500e-01   3.3308e-03   2.1317e-01
   6.2500e-02   8.4567e-04   2.1649e-01
   3.1250e-02   2.1150e-04   2.1657e-01
   1.5625e-02   5.2879e-05   2.1659e-01
   7.8125e-03   1.3221e-05   2.1662e-01
   3.9062e-03   3.3054e-06   2.1662e-01
   1.9531e-03   8.2635e-07   2.1662e-01
   9.7656e-04   2.0659e-07   2.1662e-01
   4.8828e-04   5.1647e-08   2.1662e-01
   2.4414e-04   1.2910e-08   2.1659e-01
```

```
p = (1:12)';
n = 2.^p;
dispvars = zeros(length(p),3);

for i = 1:length(p)
    N = n(i);   h = 1/N;  x = h*(0:N)';

    u_h = (x.^2-x).*exp(x);    % true solution u(0)=u(1)=0
    f_h = -(x.^2+3*x).*exp(x); % -u'' = f

    a = 2*ones(N-1,1)/(h^2);   % create a, b & c
    b = -ones(N-1,1)/(h^2);
    c = -ones(N-1,1)/(h^2);
    ftil = f_h(2:N);          % rhs is f evaluated at n-1 interior pts
```

```
   v_h = trisolve(a,b,c,ftil);  %  solve
   v_h = [0;v_h;0];             %  set BCs u(0)=u(1)=0

   dispvars(i,1) = h;
   dispvars(i,2) = max(abs(v_h-u_h));
   dispvars(i,3) = dispvars(i,2)/h^2;
end

format short e
disp(' ')
disp('       h        inf_error   inf_error/h^2')
disp('----------------------------------------')
disp(dispvars)
```