MATH 552 Scientific Computation II Spring 2016

Homework Set 2 Due Monday, 22 February

1. *Orthogonality of complex exponentials*: Consider the complex exponential functions
$$\phi_n(x) = e^{-i(\frac{n\pi x}{L})} \quad \text{for} \quad -\infty < n < \infty \, .$$
Show that
$$\langle \phi_n(x), \phi_m(x) \rangle = \int_{-L}^{L} \phi_n(x)\overline{\phi_m}(x)\; dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } n = m \, , \end{cases}$$
and thus the functions are mutually orthogonal.

2. Show that the second order centered finite difference approximation to $u''(x_j)$,
$$D^2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \, ,$$
satisifes
$$u''(x_j) = D^2 u_j + O(h^2) \, ,$$
using Taylor series approximations. Here $u_j = u(x_j)$ and $u_{j\pm1} = u(x_{j\pm1})$ where $x_{j\pm1} = x_j \pm h$. Derive a concise formula for the $O(h^2)$ error term.

3. Find **by hand** the eigenvalues and eigenvectors of the following matrices
(a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4. Write a MATLAB function M-file **trisolve.m** to solve the linear system $Ax = f$ where
$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_n & a_n \end{pmatrix}$$
is a tridiagonal $n \times n$ matrix **Assume** that no partial pivoting is required. The inputs are the n-vectors a, b, c and f and returns the solution x. Its first line should read:

```
function x = trisolve(a,b,c,f)
```

Test your code with the 5×5 system with $a_i = 2$, $b_i = -1$, $c_i = -1$, and RHS $f = [1, 0, 0, 0, 1]^T$. The exact solution is $x = [1, 1, 1, 1, 1]^T$. Use MATLAB's **diary** command to save your MATLAB session output showing that your code works properly. Include a copy of both codes.

5. Consider the 2-point BVP

$$\begin{cases} -u'' = -(x^2 + 3x)e^x \\ u(0) = u(1) = 0 \end{cases}$$

(a) Show $u(x) = (x^2 - x)e^x$ is the exact solution.

(b) Write a MATLAB function M-file to solve the problem using the 2nd order centered FD scheme we discussed in class, $-D^2 v_i = f_i$, that utilizes your m-file **trisolve.m** from problem 3 above. Note that $\sigma = 0$ here. Assume a mesh size $h = 1/n$ where $n = 2^p$ for p a positive integer. For $p = 1 : 12$ present a table with the following data - column 1: h; column 2: $\|u_h - v_h\|_\infty$; column 3: $\|u_h - v_h\|_\infty / h^2$; where $h = 1/n$. What does the trend in the third column indicate? Include a copy of your code.