HW #1 Solutions

1. *Fourier Series and Orthogonality of Sines and Cosines:* The Fourier series for $f \in L^2[-\pi, \pi]$, given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \,,$$

is an expansion of the function $f(x)$ in the basis of trigonometric functions

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \ldots\} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

Recall the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \; dx \,,$$

find an expression for $\langle \phi_n(x), \phi_m(x) \rangle$ for $m, n \geq 0$. In particular, show that the basis functions are mutually orthogonal by showing that for $n \neq m$ the inner product is zero.

ANS:

Using the identity $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \sin nx \sin mx \; dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x - \cos(n+m)x \; dx \\
&= \frac{1}{2} \begin{cases} \int_{-\pi}^{\pi} 1 - 1 \; dx & n = m = 0 \\ \int_{-\pi}^{\pi} 1 - \cos 2nx \; dx & n = m \neq 0 \\ \int_{-\pi}^{\pi} \cos(n-m)x - \cos(n+m)x \; dx & n \neq m \end{cases} \\
&= \frac{1}{2} \begin{cases} 0 & n = m = 0 \\ 2\pi - \left[\frac{1}{2n} \sin 2nx \right]\Big|_{-\pi}^{\pi} & n = m \neq 0 \\ \left[\frac{1}{(n-m)} \sin(n-m)x - \frac{1}{(n+m)} \sin(n+m)x \right]\Big|_{-\pi}^{\pi} & n \neq m \end{cases} \\
&= \frac{1}{2} \begin{cases} 0 & n = m = 0 \\ 2\pi - 0 & n = m \neq 0 \\ 0 - 0 & n \neq m \end{cases} \\
&= \begin{cases} 0 & n = m = 0 \\ \pi & n = m \neq 0 \\ 0 & n \neq m \end{cases}
\end{aligned}$$

Using the identity $\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$ and following steps similar to those above:

$$\begin{aligned}
\langle \cos nx, \cos mx \rangle &= \int_{-\pi}^{\pi} \cos nx \cos mx \; dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x + \cos(n+m)x \; dx \\
&= \begin{cases} 2\pi & n = m = 0 \\ \pi & n = m \neq 0 \\ 0 & n \neq m \end{cases}
\end{aligned}$$

Using the identity $\sin a \cos b = \frac{1}{2}(\sin(a-b) + \sin(a+b))$. Note below that if $n = m = 0$ or $m = 0$ and $n > 0$ the inner product is easily seen to be 0. Thus, assuming $m, n > 0$

$$\begin{aligned}
\langle \sin nx, \cos mx \rangle &= \int_{-\pi}^{\pi} \sin nx \cos mx \; dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x + \sin(n+m)x \; dx \\
&= \begin{cases} \int_{-\pi}^{\pi} 0 + \sin 2nx \; dx & n = m \\ -\left[\frac{1}{(n-m)} \cos(n-m)x + \frac{1}{(n+m)} \cos(n+m)x \right]\Big|_{-\pi}^{\pi} & n \neq m \end{cases} \\
&= 0.
\end{aligned}$$

2. *Separation of Variables:* Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with Neumann no flux boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad (t > 0) \\ \text{IC:} \quad & u(x,0) = f(x) \quad (t = 0) \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = G(t)\phi(x)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$.

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

(d) Write the solution in terms of an infinite series with coefficients a_n, and derive a formula for the a_n in terms of an integral involving the intial condition $u(x,0) = f(x)$.

<u>**ANS**</u>: To solve using separation of variables, we proceed as follows:

(a) Letting $u(x,t) = G(t)\phi(x)$ and substituting this expression into the PDE gives

$$G'(t)\phi(x) = \kappa G(t)\phi''(x) \quad \text{or} \quad \frac{G'(t)}{\kappa G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

where λ is the separation constant. Therefore $G(t)$ satisfies the ODE

$$G'(t) = -\lambda\kappa G(t),$$

and $\phi(x)$ the ODE with BC

$$\begin{cases} \phi''(x) = -\lambda\phi(x) \\ \phi'(0) = \phi'(L) = 0. \end{cases}$$

(b) The ODE for $G(t)$ has the general solution

$$G(t) = Ce^{-\kappa\lambda t},$$

while the solution $\phi(x)$ depends on λ:

<u>$\lambda < \mathbf{0}$</u>: General solution $\phi(x) = c_1 \cosh\sqrt{-\lambda}x + c_2 \sinh\sqrt{-\lambda}x$, and applying BC

$$0 = \phi'(0) = \sqrt{-\lambda}\,(c_1 \sinh 0 + c_2 \cosh 0) = \sqrt{-\lambda}\,(c_1{*}0 + c_2{*}1) = \sqrt{-\lambda}\,c_2 \quad \Rightarrow \quad c_2 = 0,$$

and

$$0 = \phi'(L) = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}L \quad \Rightarrow \quad c_1 = 0 \text{ since } \sinh\sqrt{-\lambda}L \neq 0.$$

Thus for $\lambda < 0$ we have only the trivial solution $\phi \equiv 0$.

$\underline{\lambda = \mathbf{0}}$: General solution $\phi(x) = c_1 + c_2 x$, and applying BC

$$0 = \phi'(0) = \phi'(L) = c_2 \quad \Rightarrow \quad c_2 = 0.$$

Thus for $\lambda = 0$ we have $\phi = c_1 = A_0$ where A_0 is an arbitrary nonzero number, which we can set to 1.

$\underline{\lambda > \mathbf{0}}$: General solution $\phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$, and applying BC

$$0 = \phi'(0) = \sqrt{\lambda}\,(-c_1\sin 0 + c_2\cos 0) = \sqrt{\lambda}\,(-c_1{*}0 + c_2{*}1) = \sqrt{\lambda}\,c_2 \quad \Rightarrow \quad c_2 = 0,$$

and

$$0 = \phi'(L) = -\sqrt{\lambda}\,c_1\sin\sqrt{\lambda}L \quad \Rightarrow \quad c_1 = 0 \text{ or } \sqrt{\lambda}L = n\pi \text{ for } n = 1, 2, 3, \ldots$$

Since $c_1 = 0$ gives the trivial solution, we take the latter, and have for $\lambda > 0$

$$\lambda_n = \frac{n\pi}{L}^2, \quad \phi_n(x) = A_n\cos\frac{n\pi x}{L} \text{ for each } n = 1, 2, 3, \ldots$$

(c) The eigenfunctions of the spatial eigenvalue-eigenfunction problem are given by

$$\{\phi_0, \phi_1, \phi_2, \ldots\} = \{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots\} = \{\phi_n(x)\}_{n=0}^{\infty}.$$

Using the results of problem 2, we have for $n, m \geq 0$

$$\begin{aligned} \langle\phi_n(x), \phi_m(x)\rangle &= \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \\ &= \frac{1}{2}\int_0^L \cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\,dx \\ &= \begin{cases} L & n = m = 0 \\ L/2 & n = m \neq 0 \\ 0 & n \neq m \end{cases} \end{aligned}$$

showing that when $n \neq m$ that $\langle\phi_n(x), \phi_m(x)\rangle = 0$, i.e. the set of eigenfunctions $\{\phi_n(x)\}_{n=0}^{\infty}$ is mutually orthogonal.

(d) Using the orthogonality of the ϕ_n, we have

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\, e^{-\kappa(\frac{n\pi}{L})^2 t}\cos\frac{n\pi x}{L}$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

3. *Inhomogeneous Boundary Conditions:* Consider the 1D heat conduction problem for $u(x,t)$ with fixed temperature boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & u(0,t) = \alpha, \quad u(L,t) = \beta, \quad (t > 0) \\ \text{IC:} \quad & u(x,0) = f(x) \quad (t = 0) \end{aligned}$$

Let $h(x) = \alpha + (\beta - \alpha)x/L$ and $v(x,t) = u(x,t) - h(x)$. Determine the PDE, BCs and IC that $v(x,t)$ satisfies. Given the solution $v(x,t)$ of the new system, explain the steps that you would go about to solve the original inhomogeneous problem for $u(x,t)$.

ANS: Let $v(x,t) = u(x,t) - h(x)$. Then

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{since} \quad \frac{d}{dt}h(x) = \frac{d^2}{dx^2}h(x) = 0,$$

and note that $h(0) = \alpha$ and $h(L) = \beta$ giving $v(0,t) = v(L,t) = 0$. Then the system for $v(x,t)$ is given by

$$\begin{cases} \dfrac{\partial v}{\partial t} = \kappa \dfrac{\partial^2 v}{\partial x^2} & (0 < x < L, t > 0) \\ v(0,t) = v(L,t) = 0 & (t > 0) \\ v(x,0) = u(x,0) - h(x) = f(x) - h(x) = g(x) & (t = 0) \end{cases}$$

Thus to solve the original system for $u(x,t)$, first solve the system for $v(x,t)$ and one has $u(x,t) = v(x,t) + h(x)$.

4. *Self-Adjointness of the Laplacian:* Consider the vector space $V = C^2([a,b])$, the set of all real-valued functions $f(x)$ defined on the interval $[a,b]$ which are at least two times continuously differentiable. Let the inner product on V be defined in the usual manner, $\langle f, g\rangle = \int_a^b f(x)g(x)\ dx$. Show that for any $f, g \in V$ that satisfy the boundary conditions

   (a) $f(a) = f(b) = 0$, and similarly for g (Dirichlet BC).

   (b) $f'(a) = f'(b) = 0$, and similarly for g (Neumann BC).

   (c) $f(a) = f(b)$, $f'(a) = f'(b)$, and similarly for g (periodic BC).

   that, considering each case above **separately**,

$$\langle \frac{d^2}{dx^2} f, g\rangle = \langle f, \frac{d^2}{dx^2} g\rangle \,,$$

   This shows that the 1D Laplacian, considered as a linear operator on V along with each of the above boundary conditions, is a *self-adjoint* operator. (Hint: In each case, use integration by parts for definite integrals *twice* and apply the boundary conditions.)

   ANS: Now, integrating by parts twice, and using the fact that the boundary terms that arise in each of the two steps is 0 for each of the BCs listed above, we have

$$\begin{aligned}
\langle \frac{d^2}{dx^2} f, g\rangle &= \int_a^b f''(x)g(x)\ dx \quad \text{and integrating by parts} \\
&= g(x)f'(x)\Big|_a^b - \int_a^b f'(x)g'(x)\ dx \\
&= -\int_a^b f'(x)g'(x)\ dx \quad \text{and integrating by parts} \\
&= -\left[\, g'(x)f(x)\Big|_a^b - \int_a^b f(x)g''(x)\ dx \,\right] \\
&= \int_a^b f(x)g''(x)\ dx \\
&= \langle f, \frac{d^2}{dx^2} g\rangle
\end{aligned}$$