MATH 552 **Scientific Computing II** **Spring 2016**

Homework Set 1 Due Friday, 5 Febuary 2016

1. *Fourier Series and Orthogonality of Sines and Cosines:* The Fourier series for $f \in L^2[-\pi,\pi]$, given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \,,$$

is an expansion of the function $f(x)$ in the basis of trigonometric functions

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \ldots\} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

Recall the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \; dx \,,$$

find an expression for $\langle \phi_n(x), \phi_m(x) \rangle$ for $m, n \geq 0$. In particular, show that the basis functions are mutually orthogonal by showing that for $n \neq m$ the inner product is zero. (Hint: It may be useful to express each trigonometric function in complex form.)

2. *Separation of Variables:* Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with Neumann no flux boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad (t > 0) \\ \text{IC:} \quad & u(x,0) = f(x) \quad (t = 0) \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = G(t)\phi(x)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$.

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

(d) Write the solution in terms of an infinite series with coefficients a_n, and derive a formula for the a_n in terms of an integral involving the intial condition $u(x,0) = f(x)$.

3. *Inhomogeneous Boundary Conditions:* Consider the 1D heat conduction problem for $u(x,t)$ with fixed temperature boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < L, t > 0) \\ \text{BCs:} \quad & u(0,t) = \alpha, \quad u(L,t) = \beta, \quad (t > 0) \\ \text{IC:} \quad & u(x,0) = f(x) \quad (t = 0) \end{aligned}$$

Let $h(x) = \alpha + (\beta - \alpha)x/L$ and $v(x,t) = u(x,t) - h(x)$. Determine the PDE, BCs and IC that $v(x,t)$ satisfies. Given the solution $v(x,t)$ of the new system, explain the steps that you would go about to solve the original inhomogeneous problem for $u(x,t)$.

4. *Self-Adjointness of the Laplacian:* Consider the vector space $V = C^2([a,b])$, the set of all real-valued functions $f(x)$ defined on the interval $[a,b]$ which are at least two times continuously differentiable. Let the inner product on V be defined in the usual manner, $\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$. Show that for any $f, g \in V$ that satisfy the boundary conditions

   (a) $f(a) = f(b) = 0$, and similarly for g (Dirichlet BC).

   (b) $f'(a) = f'(b) = 0$, and similarly for g (Neumann BC).

   (c) $f(a) = f(b)$, $f'(a) = f'(b)$, and similarly for g (periodic BC).

   that, considering each case above **separately**,

$$\langle \frac{d^2}{dx^2} f, g \rangle = \langle f, \frac{d^2}{dx^2} g \rangle \, ,$$

   This shows that the 1D Laplacian, considered as a linear operator on V along with each of the above boundary conditions, is a *self-adjoint* operator. (Hint: In each case, use integration by parts for definite integrals *twice* and apply the boundary conditions.)