

Homework Set 6 hfill Due Wednesday, 26 June 2019

1. Approximate the following definite integrals using the composite Trapezoidal rule $T(h)$ for $N = 2, 4, 8, 16, 32$ and 64 , where $h = (b - a)/N$.

(a) $\int_0^1 3x + 1 \, dx$

(b) $\int_0^1 xe^{-x^2} \, dx$

(c) $\int_0^{2\pi} \cos x + 1 \, dx$

To do so, write an M-file *mytrap.m*, the first line of which should be

```
function y = mytrap(f,a,b,N)
```

Include a copy of your code. For each of the functions above make a table, as was done in class, with columns for N , h , $T(h)$, $|\text{error}|$, and $|\text{error}|/h^2$. Are the numbers in the last column converging, and if so, what does it mean? Explain precisely the behavior of the error for (a) and (b)? Can you explain why the results for (c) are so accurate?

2. (Corrected Trapezoidal Rule) Recall that one form of the error term for the composite Trapezoidal rule $T(h)$ is

$$(*) \quad E(h) = -\frac{h^3}{12} \sum_{i=0}^{N-1} f''(\eta_i) = -\frac{h^2}{12} \left(\sum_{i=0}^{N-1} f''(\eta_i)h \right),$$

where $\eta_i \in [x_i, x_{i+1}]$, $x_i = a + ih$, and $h = (b - a)/N$. Note that the last term in the parentheses above in (*) can be viewed as a Riemann Sum approximation of

$$\int_a^b f''(x)dx = f'(b) - f'(a).$$

This suggests we could *correct* (or, more precisely improve) the accuracy of the Trapezoidal Rule by including this term if the values of $f'(a)$ and $f'(b)$ are available. The resulting numerical integration rule is called the Corrected Trapezoidal Rule, whose composite form is given by

$$CT(h) = \frac{h}{2} (f(x_0) + 2f(x_1) + \dots + 2f(x_{N-1}) + f(x_N)) - \frac{h^2}{12} (f'(b) - f'(a)).$$

It can be shown that the error for $CT(h)$ when approximating $\int_a^b f(x)dx$ is proportional to h^4 , a significant improvement over composite $T(h)$, whose error is proportional to h^2 .

- (a) Write a MATLAB function M-file to compute $CT(h)$, where the first line is of the form:

```
function y = myctrap(f,a,b,N,fpa,fpb)
```

Include a copy of your code.

(b) To numerically verify the order of $CT(h)$ apply your code to approximate

$$I = \int_0^1 x e^{-x^2} dx$$

for $N = 2, 4, 8, 16, 32, 64$. Make a table, as was done in class, with columns for N , h , $CT(h)$, $|\text{error}|$, and $|\text{error}|/h^4$. Are the numbers in the last column converging, and if so, what does it mean?

3. (*Method of Undetermined Coefficients*) We derived Simpson's rule to approximate $I(f) = \int_a^b f(x) dx$,

$$S(h) = \frac{h}{3} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right), \quad h = (b-a)/2,$$

by interpolating $f(x)$ at the points $x = a, \frac{a+b}{2}, b$, then integrating the interpolant over $[a, b]$. The approximation satisfies

$$I(f) = S(h) - \frac{1}{90} h^5 f^{(4)}(\eta), \quad \text{where } \eta \in (a, b).$$

Note that the term $f^{(4)}(\eta)$ implies Simpson's rule is *exact* if $f(x)$ is a polynomial of degree ≤ 3 , i.e., $p_n(x)$ for $0 \leq n \leq 3$.

(a) Use the method of undetermined coefficients to derive $S(h)$. Assume that

$$S(h) = c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b),$$

where the coefficients c_1, c_2 , and c_3 are to be determined. Evaluate the expression using the *three* functions $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$, and for each compute the exact answer. Derive and solve a 3×3 linear system for the coefficients.

(b) Now use $S(h)$ to approximate $I(f)$ with $f(x) = x^3$. Is the answer exact? Discuss.