Homework Set 5 Due Friday, 21 June 2019

1. Matrix norms

   (a) Consider the matrix,
   $$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$
   Compute $\|A\|_\infty$ and find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$.

   (b) Find an example of a 2×2 matrix A such that $\|A\|_\infty = 1$ but $\rho(A) = 0$. This shows that the spectral radius $\rho(A)$ **does not** define a matrix norm.

2. Consider the matrix, right side vector, and two approximate solutions,
   $$A = \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix}, \ b = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix}, \ x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ x_2 = \begin{bmatrix} 0.9911 \\ -0.4870 \end{bmatrix}.$$

   (a) Show that $x = [2, -2]^T$ is the exact solution of $Ax = b$.

   (b) Compute the error and residual vectors for x_1 and x_2.

   (c) Use MATLAB to find $||A||_\infty, ||A^{-1}||_\infty$, and $\kappa_\infty(A)$.

   (d) In class we proved a theorem relating the condition number of A, the relative error, and the relative residual. Check this result for the two approximate solutions x_1 and x_2 (using the $\infty-$ norm).

3. The following data are taken from a polynomial $P(x)$ of degree ≤ 5. What is the degree of $P(x)$? Explain how you know.

x	-2	-1	0	1	2	3
$P(x)$	-5	1	1	1	7	25

4. This problem emphasizes the difficulty of computing interpolating polynomials in standard form. The matrices which arise are called *Vandermonde* matrices, and can be quite ill-conditioned. Consider the polynomial of degree n that interpolates a set of data $\{F_i\}$ at the points $\{x_i\}$ in the form
   $$P_n(x) = \sum_{i=0}^{n} a_i x^i$$
   and has the property that
   $$P_n(x_i) = F_i.$$

   (a) Write down the system (for a general n) that must be solved to compute the a_i.

(b) Compute the coefficients for the polynomial that interpolates the function

$$F(x) = \sum_{i=0}^{n} x^i$$

at the $n+1$ points $x_i = 1 + \frac{i}{n}$ for $i = 0, \ldots, n$ (i.e. $F_i = F(x_i)$). Do this for $n = 7, 9, 11, 13$. The MATLAB code for $n = 7$ is

```
>> format long e
>> n=7;
>> x=(1+(0:1/n:1))';
>> V=fliplr(vander(x));
>> F = sum(V,2);
>> a = V\F;
```

You should use the help in MATLAB to make sure you understand what each of these commands. Note that fliplr is used so that V corresponds to the the matrix from part (a), as well as the definition of V given in class last term.

(c) Compute the ∞-norm condition number of each Vandermode matrix V in part (b). Use MATLAB's *cond* command for this.

(d) Compute the relative error in the ∞-norm of the computed coefficients a_i. The true answer is given by the n^{th} degree polynomial with $a_i = 1$ for $i = 0, \ldots, n$. Discuss the results in lieu of the condition numbers from (c).

5. Let $f(x) = 1/x$. Take $x_0 = 2$, $x_1 = 3$, $x_2 = 4$.

(a) Find the standard form of the interpolating polynomial $P_2(x)$ of $f(x)$ at the given interpolation points.

(b) Use the theorem proven in class to find an upper bound for the error

$$||f - P_2||_\infty = \max_{2 \le x \le 4} |f(x) - P_2(x)|.$$

(c) Find $||f - P_2||_\infty$ to at least 5 decimal places of accuracy.