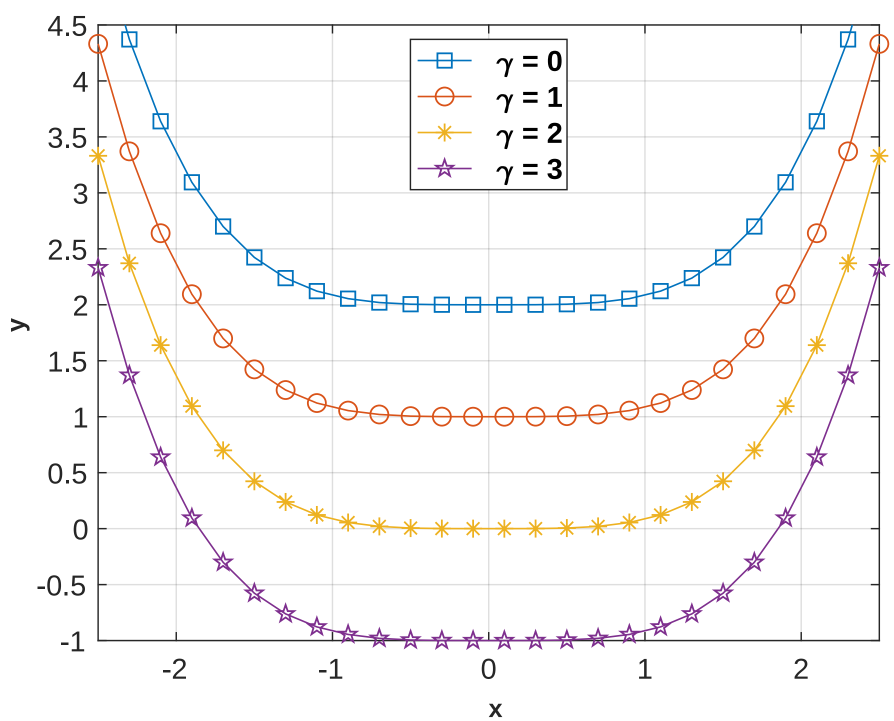
Solutions: Homework Set 3

1. Given the function $f(x) = \cosh x + \cos x - \gamma$, for each $\gamma = 1, 2, 3$ determine if there are any roots (plotting may help here!) for which one can use the bisection method to find. If so, find an interval that contains a root, and then compute it using the bisection method with a tolerance of $1e{-}10$.

   ANS: Below is the plot of $f(x)$ for each value of γ. It is clear the for $\gamma = 1$ there are no real roots. For $\gamma = 2$, there is a root of multiplicity ≥ 2 at $x = 0$ since $f(0) = \cosh 0 + \cos 0 - 2 = 1 + 1 - 2 = 0$, and $f'(0) = \sinh 0 - \sin 0 = 0 - 0 = 0$, and the function does not change sign near $x = 0$. Thus, there is no interval over which $f(x)$ changes sign. However, for $\gamma = 3$ we can see that there are two roots, of the form $x = \pm\alpha$ since $f(x)$ is an even function.



So for $\gamma = 3$ we use *my_bisect* to estimate the positve root, taking $[a, b] = [1.75, 2]$.

```
>> [root] = my_bisect('cosh(x)+cos(x)-3',1.75,2,1E-10,100)

        x_n                        f(x_n)
    ----------------------------------------------
     1.875000000000000e+00     3.755353739794653e-02
     1.812500000000000e+00    -9.486303597334933e-02
     1.843750000000000e+00    -3.036814157462109e-02
     1.859375000000000e+00     3.156614216319742e-03
     1.851562500000000e+00    -1.371381071232225e-02
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     1.855468750000000e+00    -5.305731306345596e-03
     1.857421875000000e+00    -1.081357010183304e-03
     1.858398437500000e+00     1.035927084317656e-03
     1.857910156250000e+00    -2.314010470660932e-05
     1.858154296875000e+00     5.062871746299713e-04
     1.858032226562500e+00     2.415469598848752e-04
     1.857971191406250e+00     1.091967842845598e-04
     1.857940673828125e+00     4.302667902145174e-05
     1.857925415039062e+00     9.942871972867806e-06
     1.857917785644531e+00    -6.598720161843374e-06
     1.857921600341797e+00     1.672049956269461e-06
     1.857919692993164e+00    -2.463341590264179e-06
     1.857920646667480e+00    -3.956474392552423e-07
     1.857921123504639e+00     6.382008530536609e-07
     1.857920885086060e+00     1.212766056468695e-07
     1.857920765876770e+00    -1.371854416731821e-07
     1.857920825481415e+00    -7.954424674494476e-09
     1.857920855283737e+00     5.666108915391987e-08
     1.857920840382576e+00     2.435333223971270e-08
     1.857920832931995e+00     8.199453560564507e-09
     1.857920829206705e+00     1.225148871242254e-10
     1.857920827344060e+00    -3.915955115729730e-09
     1.857920828275383e+00    -1.896720114302752e-09
     1.857920828741044e+00    -8.871028356338684e-10
     1.857920828973874e+00    -3.822937522102166e-10
     1.857920829090290e+00    -1.298894325429956e-10
     1.857920829148497e+00    -3.687716798594920e-12

root =

     1.857920829148497e+00
```

Notice that values in the $f(x_n)$ column do not go to 0 monotonically, indicating that for the bisection method the estimates x_n may get close to α and then move away slightly before continuing towards α.

2. Which of the following iterations $x_{n+1} = g(x_n)$, provided x_0 is sufficiently close to α, converge to the indicated fixed point α? Explain.

   (a) $x_{n+1} = g(x_n) = -16 + 6x_n + \dfrac{12}{x_n}, \qquad \alpha = 2$

   (b) $x_{n+1} = g(x_n) = \dfrac{2}{3}x_n + \dfrac{1}{x_n^2}, \qquad \alpha = 3^{1/3}$

   (c) $x_{n+1} = g(x_n) = \dfrac{12}{1 + x_n}, \qquad \alpha = 3$

   ANS: For each we check to make sure α is a fixed point, then examine the derivatives of $g(x)$ at α to determine the order of convergence. We want to find the order p, and in addition C in the case of linear convergence ($p = 1$), such that for n sufficiently large we have

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p.$$

   (a) $g(x) = -16+6x+12x^{-1}$, and $g(2) = -16+6*2+12*(2)^{-1} = -16+12+6 = 2$, so $\alpha = 2$ is a fixed point. $g'(x) = 6 - 12x^{-2}$, and $g'(2) = 6 - 3 = 3$. Since $|g'(2)| = |g'(\alpha)| = 3 \geq 1$, in general, given x_0 sufficiently close to α the iteration is **not** guaranteed converge.

   (b) $g(x) = 2x/3 + x^{-2}$, and $g(3^{1/3}) = 2 * 3^{-2/3} + 3^{-2/3} = 3 * 3^{-2/3} = 3^{1/3}$, so $\alpha = 3^{1/3}$ is a fixed point. $g'(x) = 2/3 - 2x^{-3}$, and $g'(3^{1/3}) = 2/3 - 2/3 = 0$, indicating that the convergence is at least quadratic. Now, $g''(x) = 6x^{-4}$, and $g''(3^{1/3}) = 6 * 3^{-4/3} \neq 0$. So indeed, if x_0 is chosen sufficeintly close to α, the order of convergence is **quadratic**, i.e. $p = 2$.

   (c) $g(x) = 12/(1 + x)$, and $g(3) = 12/(1 + 3) = 3$, so $\alpha = 3$ is a fixed point. $g'(x) = -12/(1 + x)^2$, and $g'(3) = -12/(1 + 3)^2 = -3/4$. Since $g'(\alpha) \neq 0$ but $|g'(\alpha)| < 1$, if x_0 is chosen sufficiently close to α the iteration will converge with $p = 1$, or **linearly**, with **rate** $C \approx 3/4$.

3. Consider Newton's method for finding the $\sqrt{a}$ by finding the positive root of $f(x) = x^2 - a = 0$. Assuming $x_0 > 0$ and $x_0 \neq \sqrt{a}$, show the following:

   (a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

   (b) $x_{n+1}^2 - a = \left(\dfrac{x_n^2 - a}{2x_n}\right)^2$ for $n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n \geq 1$.

   (c) The iterates $\{x_n\}_{n=0}^{\infty}$ are a strictly decreasing sequence for $n \geq 1$. *Hint*: Consider the sign of $x_{n+1} - x_n$.

   (d) A fundamental result concerning the convergence of sequences of real numbers is that if the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and monotonic, then it **converges to a finite limit**. In light of (a)-(c), discuss the convergence of Newton's method for finding $\sqrt{a}$.

ANS: To find $\sqrt{a}$ we use Newton's method to find the positive root of $f(x) = x^2 - a = 0$.

(a) Applying Newton's method to $f(x)$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Using the result from (a)

$$x_{n+1}^2 - a = \frac{1}{4}\left(x_n + \frac{a}{x_n}\right)^2 - a = \frac{1}{4}\left(x_n^2 + 2a + \frac{a^2}{x_n^2}\right) - a = \frac{1}{4}\left(x_n^2 - 2a + \frac{a^2}{x_n^2}\right) = \left(\frac{x_n^2 - a}{2x_n}\right)^2.$$

Thus, if $x_n^2 \neq a$ then $\left(\frac{x_n^2 - a}{2x_n}\right)^2 > 0$, or $x_{n+1}^2 > a$, or $x_{n+1} > \sqrt{a}$ for $n \geq 0$. It then follows that $x_n > \sqrt{a}$ for $n \geq 1$.

(c) Note using (a)

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{a}{x_n} - x_n\right) = \frac{1}{2}\left(\frac{a - x_n^2}{x_n^2}\right).$$

From (b) we have $x_n > \sqrt{a}$ for $n \geq 1$, or $a - x_n^2 < 0$. Thus the term $\left(\frac{a - x_n^2}{x_n^2}\right)$ is strictly negative for $n \geq 1$. Hence, $x_{n+1} - x_n < 0$ for $n \geq 1$, showing the sequence $\{x_n\}_{n=0}^{\infty}$ is strictly decreasing for $n \geq 1$.

(d) From x_1 on we have a **monotonic** and **bounded** sequence of real numbers, which converges by the Monotonic Sequence Theorem.

4. (Ill-behaved Root Finding) In our analysis of Newton's method we showed that if $f'(\alpha) \neq 0$ (α is a simple, multiplicity 1 root), then second order convergence results provided x_0 is chosen sufficiently close to α. However, if **α is a root of multiplicity $p \geq 2$** of $f(x)$, then it follows that

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0\,.$$

Then there exists a function $h(x)$ such that

$$f(x) = (x-\alpha)^p h(x)$$

and $h(\alpha) \neq 0$.

a) Write out the iteration function $g(x)$ for Newton's method in this case (note: it will involve $h(x)$ and $h'(x)$).

b) Show that $g'(\alpha) = 1 - 1/p \neq 0$, and explain why this implies **only linear convergence** of Newton's method for a root whose multiplicity is two or greater.

<u>ANS</u>: Given $f(x)$ the associated Newton iteration function is given by

$$g(x) = x - \frac{f(x)}{f'(x)}\,.$$

(a) First note that $f'(x) = (x-\alpha)^p h'(x) + p(x-\alpha)^{p-1}h(x)$. Then

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{(x-\alpha)^p h(x)}{(x-\alpha)^p h'(x) + p(x-\alpha)^{p-1}h(x)} \\ &= x - \frac{(x-\alpha)h(x)}{(x-\alpha)h'(x) + ph(x)} \end{aligned}$$

(b) Taking the derivative of $g(x)$ gives

$$\begin{aligned} g'(x) &= 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f''(x)]^2} \\ &= 1 - \frac{(x-\alpha)^2[h'(x)]^2 + ph^2(x) - (x-\alpha)^2 h(x)h''(x)}{[(x-\alpha)h'(x) + ph(x)]^2} \end{aligned}$$

Evaluating $g'(x)$ at $x = \alpha$, we have

$$\begin{aligned} g'(\alpha) &= 1 - \frac{(0)^2[h'(\alpha)]^2 + ph^2(\alpha) - (0)^2 h(\alpha)h''(\alpha)}{[(0)h'(\alpha) + ph(\alpha)]^2} \\ &= 1 - \frac{ph^2(\alpha)}{p^2h^2(\alpha)} = 1 - \frac{1}{p} \neq 0 \quad \text{since } p \neq 1\,. \end{aligned}$$

Thus, if the root α is not a **simple** root, then $g'(\alpha) \neq 0$, which is required for convergence that is at least quadratic (recall the proof of the convergence of Newton's method for simple roots!).

5. (Aitken's Extrapolation) Consider the fixed point iteration $x_{n+1} = g(x_n)$. Once the iterates are "*close*" to the root α then $\frac{\alpha - x_{n+1}}{\alpha - x_n} \approx g'(\alpha)$ is nearly a constant (using the MVT, and assuming $g(x)$ is smooth enough), which is independent of n. In this case we can write

$$\frac{\alpha - x_{n+1}}{\alpha - x_n} \approx \frac{\alpha - x_n}{\alpha - x_{n-1}},$$

or equivalently $(\alpha - x_{n+1})(\alpha - x_{n-1}) \approx (\alpha - x_n)^2$. One can then solve this expression for α to get an improved approximation for the fixed point. If the assumption $g'(x_n) \approx$ constant is true, the approximation for α that is obtained in this way is usually a big improvement over the last x_n in the generated sequence.

This procedure is called *Aitken's extrapolation*. Given below is a table of iterates from a linearly convergent sequence $x_{n+1} = g(x_n) = x_n - (x_n^2 - 3)/2$ used to find $\sqrt{3}$. Use Aitken's extrapolation, and the last three iterates below, to obtain an improved estimate for the fixed point α.

n	x_n
0	1.8000000000
1	1.6800000000
2	1.7688000000
3	1.7044732800
4	1.7518586988
5	1.7173542484
6	1.7427014411
7	1.7241972846
8	1.7377691464
9	1.7278483432

Compare the absolute error in x_9 to that of the new approximate root found using the Aitken's extrapolation procedure above. Make sure to use *format long e* in MATLAB is order to observe the increased accuracy.

ANS: Starting from $(\alpha - x_{n+1})(\alpha - x_{n-1}) \approx (\alpha - x_n)^2$, we have the expression

$$\alpha^2 - (x_{n+1} + x_{n-1})\alpha + x_{n+1}x_{n-1} \approx \alpha^2 - 2x_n\alpha + x_n^2 .$$

Canceling the α^2 term and solving for α (a linear expression!), gives

$$\alpha \approx \frac{x_n^2 - x_{n+1}x_{n-1}}{2x_n - (x_{n+1} + x_{n-1})} .$$

With $x_{n+1} = x_9$, $x_n = x_8$ and $x_{n-1} = x_7$, from above we have the approximation

$$\alpha = 1.73203783537268 .$$

Then $|\alpha - \sqrt{3}| = 1.2972e{-}05$, while $|x_9 - \sqrt{3}| = 4.2025e{-}03$. The approximation produced using Aitken extrapolation is appreciably more accurate.

The original iteration $x_{n+1} = g(x_n) = x_n - \dfrac{(x_n^2 - 3)}{2}$ with $x_0 = 1.8$ would require a minimum of 29 iterations to achieve comparble accuracy, i.e., you would have to compute up to the iterate x_{29}.