Homework Set 3 Due Thursday, 6 June 2019

1. Given the function $f(x) = \cosh x + \cos x - \gamma$, for each $\gamma = 1, 2, 3$ determine if there are any roots (plotting may help here!) for which one can use the bisection method to find. If so, find an interval that contains a root, and then compute it using the bisection method with a tolerance of $1e{-}10$.

2. Which of the following iterations $x_{n+1} = g(x_n)$, provided x_0 is sufficiently close to α, converge to the indicated fixed point α? Explain.

   (a) $x_{n+1} = g(x_n) = -16 + 6x_n + \dfrac{12}{x_n}, \qquad \alpha = 2$

   (b) $x_{n+1} = g(x_n) = \dfrac{2}{3}x_n + \dfrac{1}{x_n^2}, \qquad \alpha = 3^{1/3}$

   (c) $x_{n+1} = g(x_n) = \dfrac{12}{1 + x_n}, \qquad \alpha = 3$

3. Consider Newton's method for finding the $\sqrt{a}$ by finding the positive root of $f(x) = x^2 - a = 0$. Assuming $x_0 > 0$ and $x_0 \neq \sqrt{a}$, show the following:

   (a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$

   (b) $x_{n+1}^2 - a = \left(\dfrac{x_n^2 - a}{2x_n}\right)^2$ for $n \geq 0$, and thus $x_n > \sqrt{a}$ for all $n \geq 1$.

   (c) The iterates $\{x_n\}_{n=0}^{\infty}$ are a strictly decreasing sequence for $n \geq 1$. *Hint*: Consider the sign of $x_{n+1} - x_n$.

   (d) A fundamental result concerning the convergence of sequences of real numbers is that if the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and monotonic, then it **converges to a finite limit**. In light of (a)-(c), discuss the convergence of Newton's method for finding $\sqrt{a}$.

4. (Ill-behaved Root Finding) In our analysis of Newton's method we showed that if $f'(\alpha) \neq 0$ (α is a simple, multiplicity 1 root), then second order convergence results provided x_0 is chosen sufficiently close to α. However, if $\boldsymbol{\alpha}$ **is a root of multiplicity** $\boldsymbol{p \geq 2}$ of $f(x)$, then it follows that

   $$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0\,.$$

   Then there exists a function $h(x)$ such that

   $$f(x) = (x - \alpha)^p h(x)$$

   and $h(\alpha) \neq 0$.

   a) Write out the iteration function $g(x)$ for Newton's method in this case (note: it will involve $h(x)$ and $h'(x)$).

b) Show that $g'(\alpha) = 1 - 1/p \neq 0$, and explain why this implies **only linear convergence** of Newton's method for a root whose multiplicity is two or greater.

5. (Aitken's Extrapolation) Consider the fixed point iteration $x_{n+1} = g(x_n)$. Once the iterates are "*close*" to the root α then $\frac{\alpha - x_{n+1}}{\alpha - x_n} \approx g'(\alpha)$ is nearly a constant (using the MVT, and assuming $g(x)$ is smooth enough), which is independent of n. In this case we can write

$$\frac{\alpha - x_{n+1}}{\alpha - x_n} \approx \frac{\alpha - x_n}{\alpha - x_{n-1}},$$

or equivalently $(\alpha - x_{n+1})(\alpha - x_{n-1}) \approx (\alpha - x_n)^2$. One can then solve this expression for α to get an improved approximation for the fixed point. If the assumption $g'(x_n) \approx$ constant is true, the approximation for α that is obtained in this way is usually a big improvement over the last x_n in the generated sequence.

This procedure is called *Aitken's extrapolation*. Given below is a table of iterates from a linearly convergent sequence $x_{n+1} = g(x_n) = x_n - (x_n^2 - 3)/2$ used to find $\sqrt{3}$. Use Aitken's extrapolation, and the last three iterates below, to obtain an improved estimate for the fixed point α.

n	x_n
0	1.8000000000
1	1.6800000000
2	1.7688000000
3	1.7044732800
4	1.7518586988
5	1.7173542484
6	1.7427014411
7	1.7241972846
8	1.7377691464
9	1.7278483432

Compare the absolute error in x_9 to that of the new approximate root found using the Aitken's extrapolation procedure above. Make sure to use *format long e* in MATLAB is order to observe the increased accuracy.