Math 551 **Introduction to Scientific Computing** **Spring 2019**

SOLUTIONS: Homework Set 1

1. Consider the polynomial $f(x) = x^2 - x - 2$.

   (a) Find $P_1(x)$, $P_2(x)$ and $P_3(x)$ for $f(x)$ about $x_0 = 0$. What is the relation between $P_3(x)$ and $f(x)$? Why?

   (b) Find $P_1(x)$, $P_2(x)$ and $P_3(x)$ for $f(x)$ about $x_0 = 2$. What is the relation between $P_3(x)$ and $f(x)$? Why?

   (c) In general, given a polynomial $f(x)$ with degree $\leq m$, what can you say about $f(x) - P_n(x)$ for $n \geq m$?

   ANS: First note that $f'(x) = 2x - 1$, $f''(x) = 2$, and $f'''(x) \equiv 0$. Then we have

   (a) Let's find $P_3(x)$ which will also gives us $P_1(x)$ and $P_2(x)$. We have for $x_0 = 0$:

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\ &= -2 + (-1)(x - 0) + \frac{2}{2}(x - 0)^2 + \frac{0}{6}(x - 0)^3 \\ &= \mathbf{-2 - x + x^2} \end{aligned}$$

   So, $P_2(x) = -2 + (-1)(x - 0) + \frac{2}{2}(x - 0)^2 = \mathbf{-2 - x + x^2}$, and $P_1(x) = -2 + (-1)(x - 0) = \mathbf{-2 - x}$. $P_3(x) = f(x)$ because $f'''(x) \equiv 0$, and thus we must have $R_3(x) \equiv 0$.

   (b) Again, we find $P_3(x)$ which also gives us $P_1(x)$ and $P_2(x)$. With $x_0 = 2$ we have:

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\ &= 0 + 3(x - 2) + \frac{2}{2}(x - 2)^2 + \frac{0}{3!}(x - 0)^3 \\ &= \mathbf{-2 - x + x^2} \end{aligned}$$

   So, $P_2(x) = 0 + 3(x - 0) + \frac{2}{2}(x - 2)^2 = \mathbf{-2 - x - x^2}$, and $P_1(x) = 0 + 3(x - 2) = \mathbf{3x - 6}$. And again, as in **(a)**, $P_3(x) = f(x)$ because $f'''(x) \equiv 0$.

   (c) We will have that $f(x) - P_n(x) \equiv 0$ since $f(x)$ is a polynomial of degree at most m, thus $f^{(n+1)}(x) \equiv 0$ when $n \geq m$, hence the error term is identically zero.

2. Find both $P_2(x)$ and $P_3(x)$ for $f(x) = \cos x$ about $x_0 = 0$, and use them to approximate $\cos(0.1)$. Show that in each case the remainder term provides an upper bound for the true error.

ANS: First note that $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f''''(x) = \cos x$. Let's find $P_3(x)$ which will also gives us $P_2(x)$. We have for $x_0 = 0$:

$$\begin{aligned} P_3(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\ &= 1 + 0(x - 0) + \frac{(-1)}{2}(x - 0)^2 + \frac{0}{6}(x - 0)^3 \\ &= \mathbf{1} - \frac{\mathbf{x^2}}{\mathbf{2}} \end{aligned}$$

Since $f^{(3)}(0) = 0$ we also have $P_2(x) = \mathbf{1 - x^2/2}$, so in this case $P_3(x) \equiv P_2(x)$. We have $P_3(0.1) = P_2(0.1) = 1 - (0.1)^2/2 = 1 - 1/200 = 199/200 = 0.995$. Since both Taylor polynomials are the same the error in both cases is

$$|\cos(0.1) - 0.995| \approx 4.165278025713981e{-}06.$$

From Taylor's theorem, the error that results from using $P_2(x)$ as an approximate (note $n = 2$) is

$$\begin{aligned} |\cos 0.1 - 0.995| &= \left| \frac{f^{(3)}(\xi_x)}{3!}(0.1 - 0)^3 \right| \quad \text{for } \xi_x \in (0, 0.1) \\ &= |\sin(\xi_x)/6000| \quad \text{for } \xi_x \in (0, 0.1) \\ &\le \sin(0.1)/6000 \\ &\approx 1.663890277447136e{-}05 \end{aligned}$$

Now, again from Taylor's Theorem, the error using $P_3(x)$ we have (note $n = 3$)

$$\begin{aligned} |\cos 0.1 - 0.995| &= \left| \frac{f^{(4)}(\xi_x)}{4!}(0.1 - 0)^4 \right| \quad \text{for } \xi_x \in (0, 0.1) \\ &= |\cos(\xi_x)/240000| \quad \text{for } \xi_x \in (0, 0.1) \\ &\le \cos(0)/240000 \\ &\approx 4.166666666666667e{-}06 \end{aligned}$$

So in each case an upper bound derived using the error term for Talyor polynomials is indeed **larger** than the actual error.

3. Consider $f(x) = e^x$, and find a general formula for the Taylor polynomial $P_n(x)$ for f about $x_0 = 0$.

   (a) Using the remainder term, find a minimum value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

   (b) Prove that $f(x)$ analytic on $(-\infty, \infty) = \mathbb{R}$.

ANS: Note that $f^{(n)}(x) = e^x$ so for $n \geq 0$, with $x_0 = 0$, we have

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{1}{k!} x^k \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \end{aligned}$$

(a) The remainder term is given by $R_n(x) = \dfrac{f^{(n+1)}(\xi_x)}{(n+1)!} x^{n+1} = \dfrac{e^{\xi_x}}{(n+1)!} x^{n+1}$ for $\xi_x \in (0, 0.5)$, so we need to find the minimum value of n such that

$$\max_{x \in [0,0.5]} |R_n(x)| = \max_{x \in [0,0.5]} \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \leq \frac{e^{1/2}}{(n+1)!} \frac{1}{2^{n+1}} \leq 10^{-6},$$

or we need the minimum n such that

$$2^{n+1}(n+1)! \geq e^{1/2} \times 10^6 \approx 1648721.270700128$$

Just trying some values of n on the right one sees that $2^7 \times 7! = 654120$ and $2^8 \times 8! = 10321920$, so with $n + 1 = 8$ we see that one must have $n \geq 7$.

(b) We need to show that for each value of $x \in \mathbb{R}$ that

$$\lim_{n\to\infty} |e^x - P_n(x)| = \lim_{n\to\infty} |f(x) - P_n(x)| = \lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{f^{n+1}(\xi_x)}{(n+1)!}(x-0)^{n+1} \right| = 0,$$

To do so, **FIX** an $x \in \mathbb{R}$ and note that there must exist a **postive** integer M (i.e., $M \in \{1, 2, 3, 4, \ldots\}$) such that $M > |x|$. Why? Because x is fixed. Suppose $x = \pm 134{,}665{,}323.33452$, take $M = 200{,}000{,}000$ if you like, or $M = 134{,}665{,}324$. Also, since ξ_x lies in the interval between x and $x_0 = 0$, then $|\xi_x| < M$. So once $n > M$,

$$\begin{aligned} |R_n(x)| &= \left| \frac{e^{\xi_x}}{(n+1)!}(x-0)^{n+1} \right| \leq \left| \frac{e^M}{(n+1)!} M^{n+1} \right| = e^M \times \frac{M * M * M * \cdots * M}{1 * 2 * 3 * \cdots * (n+1)} \\ &= e^M \times \left(\frac{M}{1} * \frac{M}{2} * \frac{M}{3} \cdots * \frac{M}{M-1} * \frac{M}{M} * \frac{M}{M+1} * \frac{M}{M+2} * \cdots * \frac{M}{(n+1)} \right) \\ &= e^M \times \left(\frac{M}{1} * \frac{M}{2} * \frac{M}{3} \cdots * \frac{M}{M-1} * \frac{M}{M} \right) * \left(\frac{M}{M+1} * \frac{M}{M+2} * \cdots * \frac{M}{(n+1)} \right) \end{aligned}$$

Note that since x is fixed the first two terms above are **bounded**. What is true about each ratio in the last term? You should be able to complete the proof from here.

4. Given a function $f(x)$, use Taylor approximations to derive a second order *one-sided* approximation to $f'(x_0)$ is given by

$$f'(x_0) = af(x_0) + bf(x_0 + h) + cf(x_0 + 2h) + O(h^2)\,.$$

What is the precise form of the error term? Using the formula approximate $f'(1)$ where $f(x) = e^x$ for $h = 1/(2^p)$ for $p = 1 : 15$. Form a table with columns giving h, the approximation, absolute error and absolute error divided by h^2. For each indicate to which values they are converging. Finally, verify that the last column appears to be converging to a value derived using the error term.

ANS: We have the following Taylor expansions:

$$\begin{aligned} f(x_0) &= f(x_0) \\ f(x_0+h) &= f(x_0) + hf'(x_0) + \tfrac{1}{2}h^2 f''(x_0) + \tfrac{1}{6}h^3 f'''(\xi_1) \quad \text{where } \xi_1 \in (x_0, x_0+h) \\ f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \tfrac{4}{3}h^3 f'''(\xi_2) \quad \text{where } \xi_2 \in (x_0, x_0+2h) \end{aligned}$$

Forming the linear combination gives: $af(x_0) + bf(x_0+h) + cf(x_0+2h) =$

$$(a+b+c)f(x_0) + (hb+2hc)f'(x_0) + (\frac{1}{2}h^2 b + 2h^2 c)f''(x_0) + \frac{1}{6}h^3 b f'''(\xi_1) + \frac{4}{3}h^3 c f'''(\xi_2)\,.$$

Since we have three unknowns a, b, and c, we choose them so that $f'(x_0)$ is multiplied by 1, and $f(x_0)$ and $f''(x_0)$ are multiplied by 0. Thus a, b, and c must satisfy

$$\begin{aligned} a+b+c &= 0 \\ hb+2hc &= 1 \\ \tfrac{1}{2}h^2 b + 2h^2 c &= 0 \end{aligned} \quad \Rightarrow \quad a = -\tfrac{-3}{2h},\ b = \tfrac{4}{2h},\ c = \tfrac{-1}{2h}\,.$$

Using these values gives us the approximation

$$f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h}\,.$$

Error term? We have

$$\frac{1}{6}h^3 b f'''(\xi_1) + \frac{4}{3}h^3 c f'''(\xi_2) = \frac{1}{3}h^2(f'''(\xi_1) - 2f'''(\xi_2)) = -\frac{1}{3}h^2 f'''(\xi), \quad \xi \in (x_0, x_0+2h)\,,$$

so as $h \to 0$, $|error/h^2|$ should approach $|\frac{1}{3}f'''(x_0)| = \frac{1}{3}e \approx 9.0609e-01$.

h	err	err/h^2
5.0000e-01	3.3543e-01	1.3417e+00
2.5000e-01	6.8607e-02	1.0977e+00
1.2500e-01	1.5566e-02	9.9623e-01
6.2500e-02	3.7103e-03	9.4983e-01
3.1250e-02	9.0590e-04	9.2764e-01
1.5625e-02	2.2383e-04	9.1679e-01
7.8125e-03	5.5629e-05	9.1142e-01
3.9062e-03	1.3866e-05	9.0875e-01

1.9531e-03	3.4615e-06	9.0742e-01
9.7656e-04	8.6475e-07	9.0676e-01
4.8828e-04	2.1611e-07	9.0644e-01
2.4414e-04	5.4019e-08	9.0629e-01
1.2207e-04	1.3512e-08	9.0679e-01
6.1035e-05	3.3896e-09	9.0989e-01
3.0518e-05	8.7578e-10	9.4036e-01

As $h \to 0$ so does the error **until** roundoff begins to creep into the calculation, which can be seen in the last few entries since the $|error/h^2|$ column approaches $e/3$ but then starts to move away from it.

5. MATLAB: Download and modify the m-file *fp_example.m* with

```
N= (1:20)';    h=2.^(-N);
```

Also, add a *title* to the graph containing **your** full name. Run the script, printout a hardcopy of the graph and hand it in.