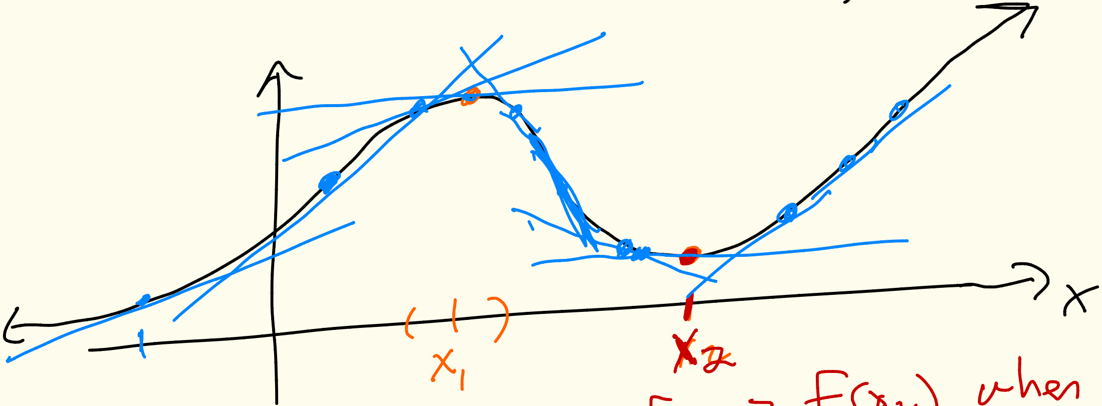
4.1: what derivatives tell us about the shape of a function?

$y = f(x)$

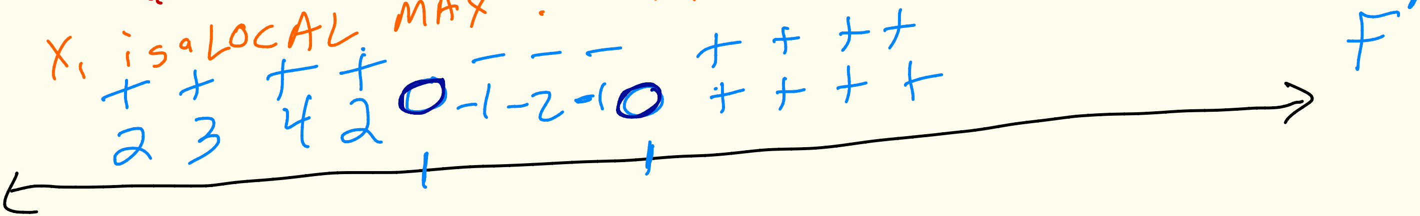


$f(x) \geq f(x_2)$ when x is "near" x_2

x_2 is a LOCAL MIN!

$f(x) \leq f(x_1)$ when x is "near" x_1

x_1 is a LOCAL MAX:



Defn: x is a CRITICAL POINT of $f(x)$ if $f'(x) = 0$ or $f'(x)$ DOES NOT EXIST.

1st Derivative Test: Suppose p is a critical point

(i) If $f'(x)$ changes from $+$ to $-$ as we pass through $p \Rightarrow$ $\boxed{p \text{ is a LOCAL MAX}}$

(ii) If $f'(x)$ changes from $-$ to $+$ as we pass through $p \Rightarrow$ $\boxed{p \text{ is a LOCAL MIN}}$

Ex: Find the LOCAL MAXs and MINs of

$f(x) = x^2 - 4x + 3 = (x-1)(x-3)$

(1) Find Critical pts

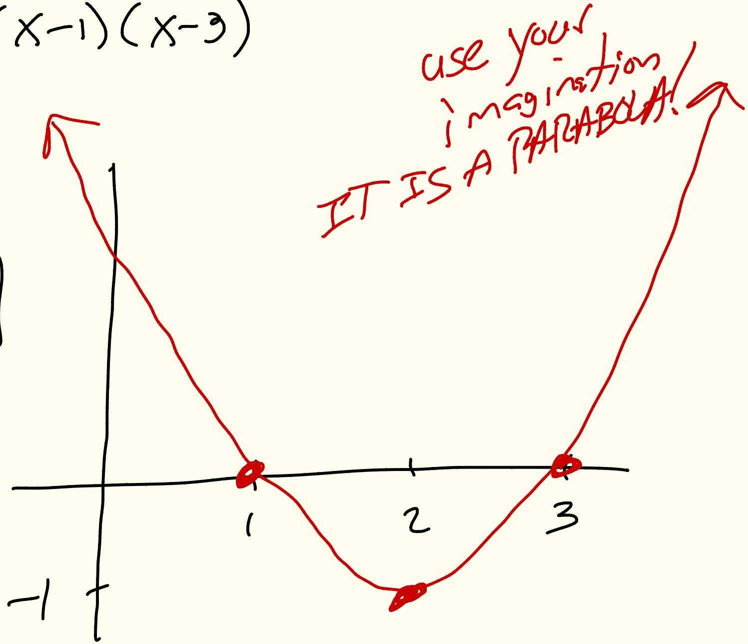
$f'(x) = 2x - 4 = 0$

$2x - 4 = 0 \Rightarrow \boxed{x = 2}$

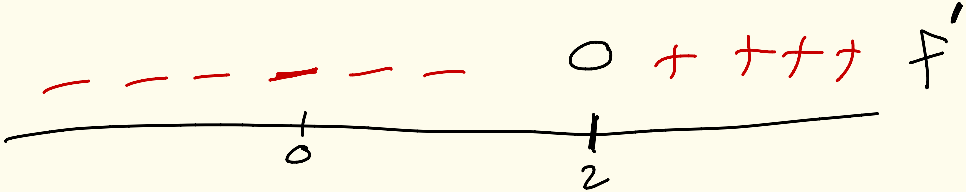
$f(2) = 4 - 8 + 3 = -1$

$f'(2) = 0, \quad f'(0) = -4 < 0$

$f'(3) = 6 - 4 = 2 > 0$



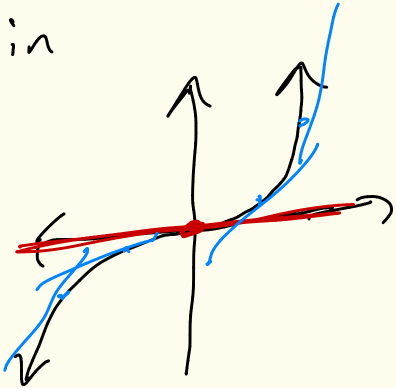
1st Derivative test TELLS us that $x=2$ is LOCAL MIN



BE CAREFUL!

If p is a LOCAL MAX or LOCAL MIN $\Longrightarrow$ p is critical pt

BUT if p is a critical p it DOES NOT have to be a local max or min



Ex: $f(x) = x^3$

$f'(x) = \boxed{3x^2} = 0$

$\Longrightarrow$ $x = 0$ is the ONLY crit. pt

BUT NOT a local max or min

+ + + + + 0 + + + + f'

0

Ex: $f(x) = x^3 - 9x^2 - 48x + 52$, Find local maxs and mins

step 1: Find the critical pts

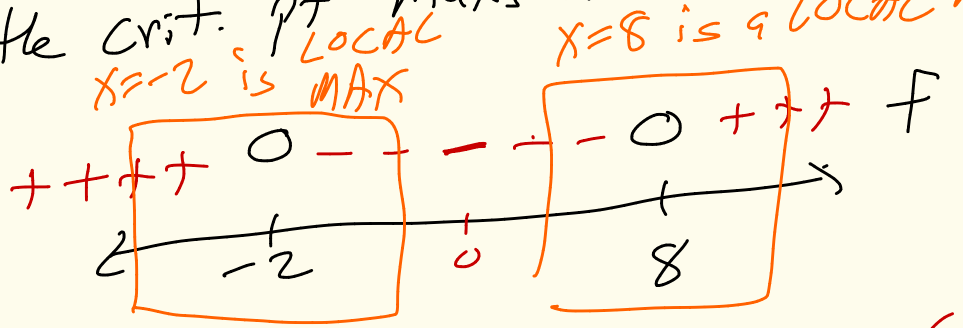
$f'(x) = \boxed{3x^2 - 18x - 48 = 0} \Rightarrow 3(x^2 - 6x - 16) = 0$

$3(x-8)(x+2) = 0$

$\Rightarrow$ $\boxed{x = -2 \text{ and } x = 8 \text{ are critical pts}}$

step 2: Are the crit. pt maxs or mins

1st Derivative Test:



$f'(0) = -48 < 0$, $f'(-10) = 3(-10-8)(-10+2) = 3\cdot(-)(-) = +$

2nd Derivative Test

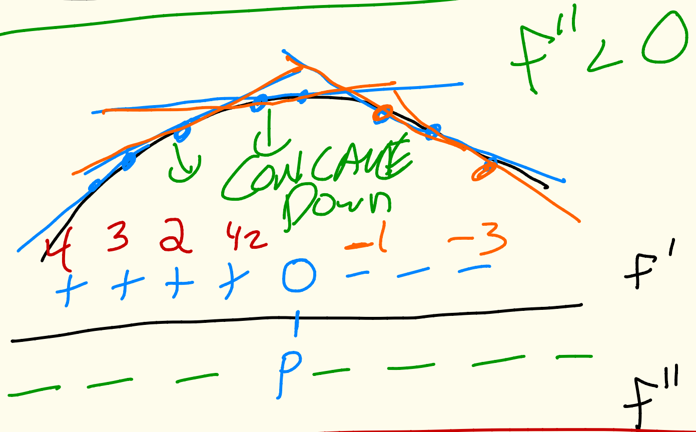
$f' > 0 \Rightarrow f$ increasing

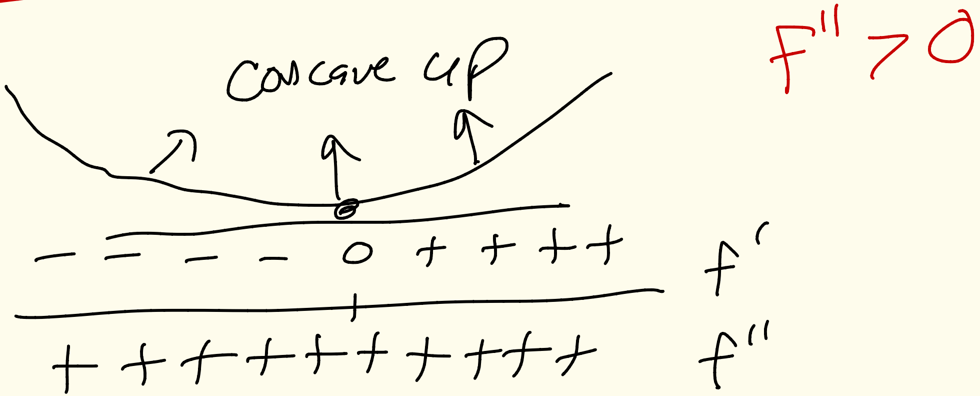
$f' < 0 \Rightarrow f$ decreasing

well

$f'' > 0 \Rightarrow f'$ is increasing

$f'' < 0 \Rightarrow f'$ is decreasing





2^{nd} Derivative Test:

Suppose p is a critical point, so $f'(p) = 0$ or $f'(p) = DNE$

(i) If $f''(p) > 0$ the the graph is CONCAVE UP at p $\Rightarrow$ p is a LOCAL MIN.

(ii) If $f''(p) < 0$ the graph is CONCAVE DOWN at p $\Rightarrow$ p is a LOCAL MAX.

(iii) If $f''(p) = 0$ $\Rightarrow$ test fails

Note: 2^{nd} Derivative Test ONLY uses information about f' and f'' at $x = p$.

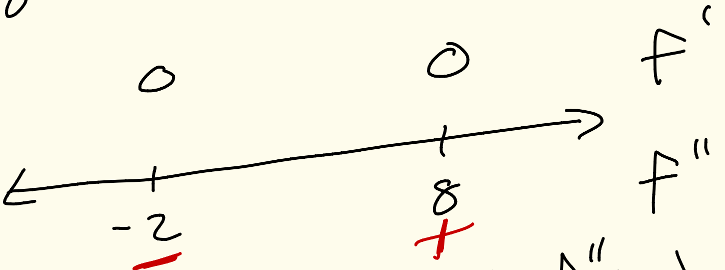
Back to the example: $f(x) = x^3 - 9x^2 - 48x + 52$

$f'(x) = 3x^2 - 18x - 48 = 0 \Rightarrow 3(x-8)(x+2) = 0$

$\Rightarrow x = -2$ and $x = 8$ crit. pts

$f''(x) = 6x - 18$

chart:



$\boxed{f''(-2) = -30} \Rightarrow f'(-2) = 0$ and $f''(-2) < 0$

so $x = -2$ is a LOCAL MAX

$f''(8) = 48 - 18 = 30 > 0 \Rightarrow f'(8) = 0$ and $f''(8) < 0$

so $x = 8$ is a LOCAL MIN

Chapter 4, Section 4.1, Question 22

The function $f(x) = x^4 - 11x^3 + 29x$ has a critical point at $x = 1$.

Use the second derivative test to identify it as a local maximum or local minimum.

Check $x=1$ is a crit. pt:

$f'(x) = 4x^3 - 33x^2 + 29$

$\boxed{f'(1) = 4 - 33 + 29 = 0}$ YES!

Max or Min?

$f''(x) = 12x^2 - 66x$

$\boxed{f''(1) = 12 - 66 = -54 < 0}$

2nd Derivative Test tell us

$f'(1) = 0$

$f''(1) < 0$

$\Longrightarrow$ $x = 1$ is a LOCAL MAX