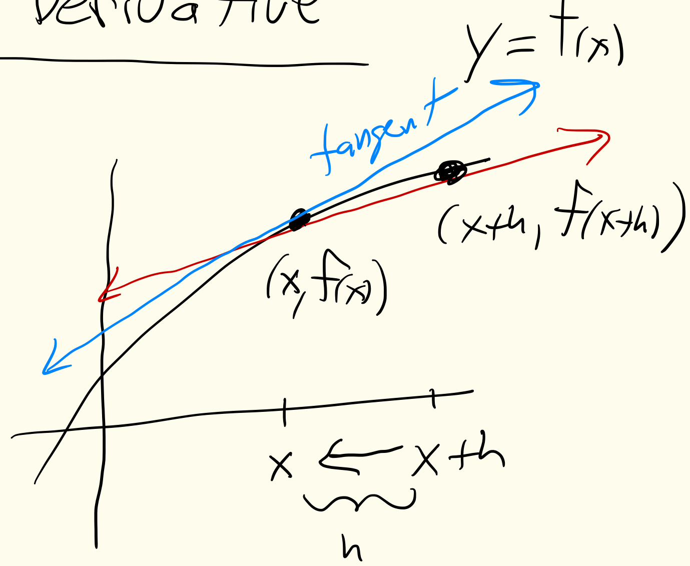
Limits, Continuity and the Derivative

what does $\lim\limits_{h\to 0}$ mean?



$h \to 0 \neq h$

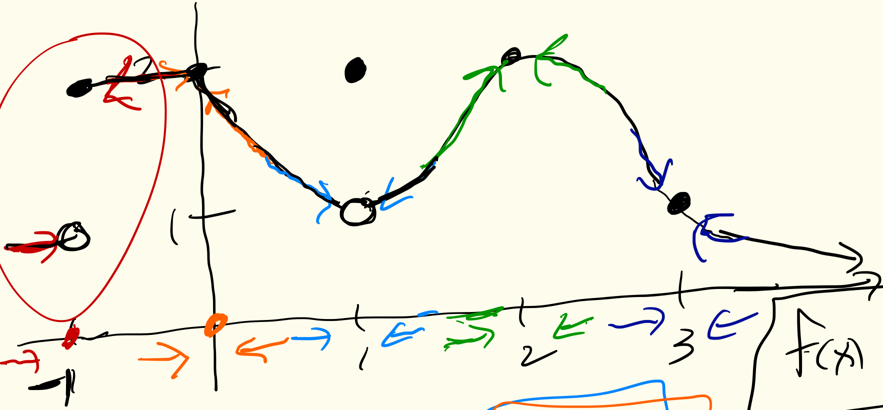
limit as h approaches 0

Defn

$$f'(x) = \boxed{\lim_{h\to 0}} \frac{f(x+h) - f(x)}{h}$$

= slope of the tangent line to the graph of $f(x)$ at x.

Limits and Continuity: $y = f(x)$



$$\lim_{x \to 0} f(x) = 2 = f(0)$$

$$\lim_{x \to -1} f(x) = \text{DNE (does not exist)}$$

$f(x)$ is continuous at 0, 2, 3

$$\lim_{x \to 1} f(x) = 1 \neq f(1) = 2$$

$$\lim_{x \to 2} f(x) = 2 = f(2) = 2$$

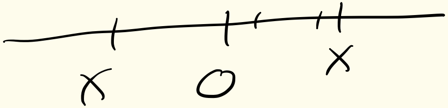
$$\lim_{x \to 3} f(x) = 1 = f(3) = 1$$

Continuity: $f(x)$ is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

(1) (3) (2)

Ex: Estimate $\displaystyle\lim_{x\to 0} \frac{2^x-1}{x} \approx 0.69314 \quad \left(\frac{0}{0}\right)$



x	$\dfrac{2^x-1}{x}$
0.1	0.7177 ....
0.01	0.6955 ....
0.001	0.6933 ....
0.0001	0.69314 ...
−0.001	0.6929 ....
−0.01	0.6907 ...
−0.1	0.6696 ....

$f(x) = 2^x \Rightarrow f'(x) = (\ln 2)2^x$

LATER!

$$\lim_{x\to 0} \frac{2^x-1}{x}$$

$$= \lim_{h\to 0} \frac{2^h-1}{h}$$

$$= \lim_{h\to 0} \frac{2^{(0+h)}-2^0}{h}$$

$$= f'(0) = \underbrace{\ln 2}_{\text{FACT}} = 0.693147\ldots$$

Ex: Estimate $\lim_{h\to 0} \frac{(3+h)^2-9}{h} \approx 6$ $\qquad \left(\frac{0}{0}\right)$?

h	$\frac{(3+h)^2-9}{h}$
0.1	6.1000...
0.01	6.0099...
0.001	6.000999...
	6?
−0.001	5.99899...
−0.01	5.9899...
−0.1	5.9..

Exact

$$(3+h)^2-9 = (3+h)(3+h)-9$$
$$= 9+6h+h^2-9$$
$$= 6h+h^2$$

$$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0}\frac{6h+h^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}} \quad \text{since } h\neq 0$$

$$\boxed{= \lim_{h\to 0} 6+h = 6+0 = \boxed{6}}$$

Big Picture

given $\boxed{f(x) \quad \text{Find} \quad \underline{f'(x)}}$ = the DERIVATIVE function

slope of tangent line at $x=2$ $= f'(2)$

slope at tangent line at $x = \pi$ $= f'(\pi)$

Ex! Find $f'(x)$ where $f(x) = x^2 \Rightarrow f'(x) = 2x$

$$f'(x) \overset{\text{defn}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\overbrace{\cancel{x^2} + 2xh + h^2}^{(x+h)^2} - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \quad \text{since } h \neq 0$$

$$= \lim_{h\to 0} 2x + h = \lim_{h\to 0} 2x + \lim_{h\to 0} h$$

$$= 2x + 0 = 2x$$

$f(x) = x^2$ so $r = 2$

$f'(x) = 2 \cdot x^{2-1} = 2x^1 = 2x$

Fact: $f(x) = x^r$

$f'(x) = r x^{r-1}$

— power rule

23. Estimate the limit: $\lim_{h\to 0} \dfrac{\frac{1}{2(3+h)} - \frac{1}{6}}{h}$ by using smaller and smaller values of h. Which of the following is the best estimate for this limit?

(A) 1666.5555

(B) -0.05555

(C) -0.0005376

(D) 1

$$\frac{\frac{1}{2(3+h)} - \frac{1}{6}}{h}$$

h	$\frac{\frac{1}{2(3+h)} - \frac{1}{6}}{h}$
0.01	$-0.055\ldots$
0.001	$-0.0555\ldots$
$\boxed{-0.001}$	$-0.05557\ldots$
-0.01	

EXACT

$$-\frac{1}{2x^2}$$

$$-\frac{1}{2\cdot 9} = -\frac{1}{18}$$

$$= -0.05555\ldots$$

24. Which of the following functions is *not* continuous on the given intervals?

(A) $f(x) = sinx$ on $-{}^{\pi}/_{2} \le x \le {}^{\pi}/_{2}$

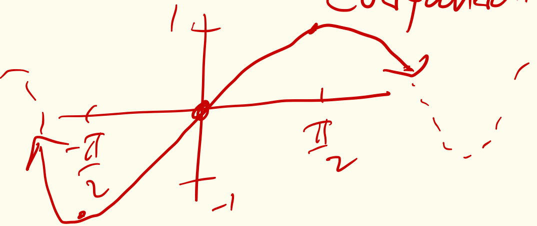
(B) $f(x) = e^{x} - 1$ on $0 \le x \le 5$

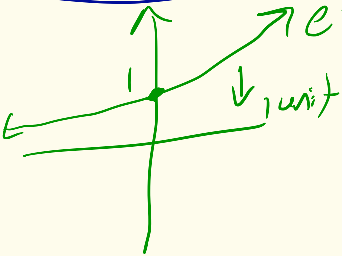
(C) $f(x) = lnx$ on $2 \le x \le 3$

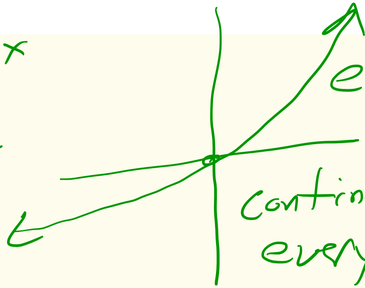
(D) $f(x) = \sqrt{2x+3}$ on $-2 \le x \le 2$

sin x

continuous everywhere!

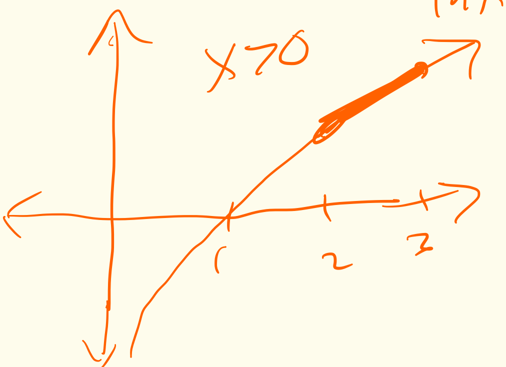






continuous everywhere

ln x

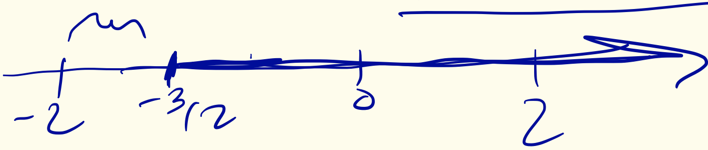


$\sqrt{2x+3} \Rightarrow 2x+3 \ge 0$

$2x \ge -3$

$\boxed{x \ge -3/2}$

not defined



25. Which of the following represents the derivative of the function $y = 6x^4$

$$F(x) = 6x^4$$

(A) $\lim\limits_{h \to 0} \frac{6(x+h)^4 - 6x^4}{x}$

(B) $\lim\limits_{x \to 0} \frac{6(x+h)^4 - 6x^4}{h}$

(C) $\lim\limits_{x \to 0} \frac{6(x+h)^4 - 6x^4}{x}$

(D) $\boxed{\lim\limits_{h \to 0} \frac{6(x+h)^4 - 6x^4}{h}}$

$$f'(x) \overset{\text{defn}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{for any } f$$

$$= \lim_{h \to 0} \frac{6(x+h)^4 - 6x^4}{h}$$