1.8: New Functions from Old

Composite Functions: given $f(t)$ and $g(t)$

$$f(g(t)) = (f \circ g)(t) \qquad\qquad g(f(t)) = (g \circ f)(t)$$

$$t \xrightarrow{g} g(t) \xrightarrow{f} f(g(t)) \qquad\qquad t \xrightarrow{f} f(t) \xrightarrow{g} g(f(t))$$

LATER:

Differentiation Rule

$$\frac{d}{dt}\left(f(g(t))\right) = f'(g(t)) \cdot g(t)$$

CHAIN RULE

Ex: $f(t) = 2t \quad g(t) = t^2+1$

a) $f(3) = 2\cdot 3 = 6$

b) $f(t+1) = 2(t+1)$
$= 2t+2$

c) $g(f(t)) = (2t)^2+1 = 4t^2+1$

d) $f(g(t)) = f(t^2+1) = 2(t^2+1) = 2t^2+2$

Note: $g(f(t)) \neq f(g(t))$ in general

e) $g(t^2) = (t^2)^2+1 = t^4+1$

f) $g(f(4)) = g(8) = 8^2+1 = 65$

$f(\square) = 2\cdot\square \qquad g(\smiley) = \smiley^2+1$

Ex:

x	0	1	2	3
$f(x)$	3	1	-1	-3
$g(x)$	0	2	4	6

a) $g(f(0)) = g(3) = 6$

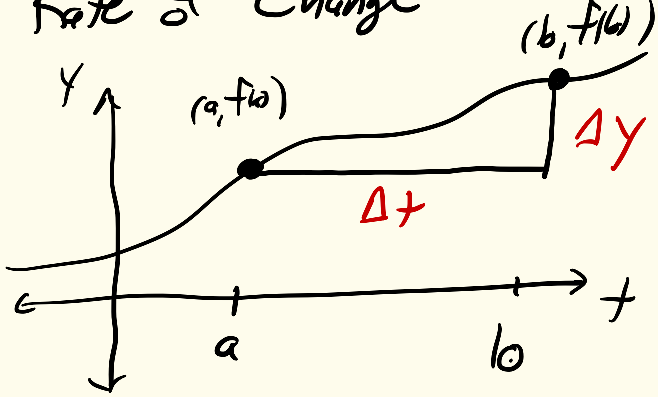
b) $f(g(0)) = f(0) = 3$

c) $g(3) = 6$

d) $f(7) = ?$ not given!

1.3: Average and Relative Rate of Change

$y = f(t)$



change

Avg Rate of Change of y between $t = a$ and $t = b$

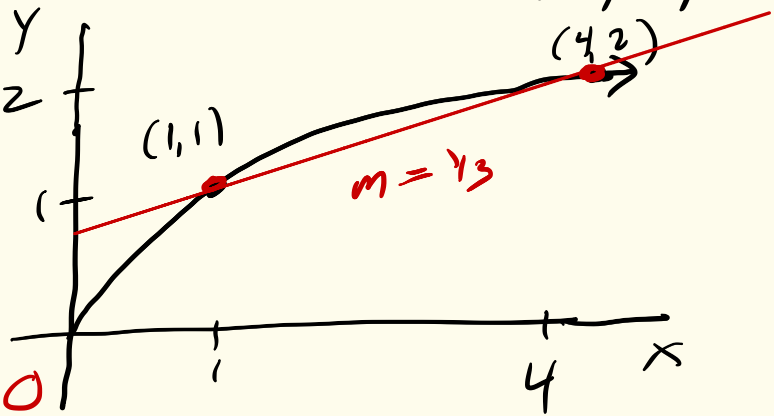
$$= \frac{\Delta y}{\Delta t} = \frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}$$

change

= slope of the line through $(a, f(a))$ and $(b, f(b))$

Ex: Find the avg rate of change of $f(x)=\sqrt{x}$ between $x=1$ and $x=4$

domain $(\sqrt{x}) = [0,\infty)$
range $(\sqrt{x}) = [0,\infty)$



ans:

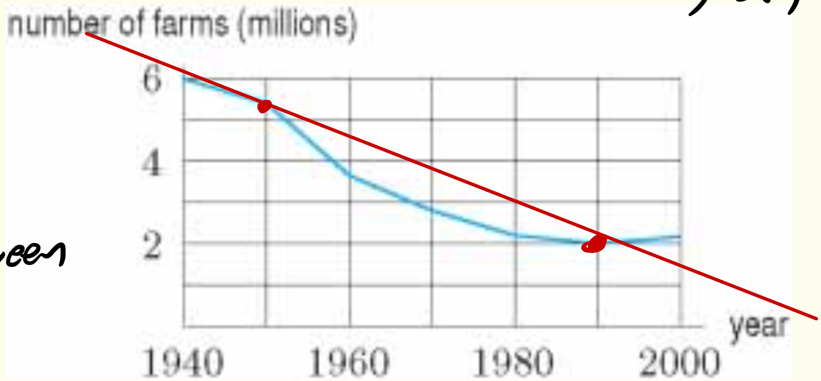
$$\frac{\Delta y}{\Delta x} = \frac{f(4)-f(1)}{4-1}$$

$$= \frac{\sqrt{4}-\sqrt{1}}{4-1} = \frac{2-1}{3} = \boxed{1/3} > 0$$

Ex: # farms in the US in millions

(1950, 5.4)
(1990, 2)

Estimate the avg rate of change of the # of farms in US between 1950 and 1990



$$\frac{\Delta N}{\Delta t} = \frac{2-5.4}{1990-1950} = \frac{-3.4}{40} = -0.085 \ \frac{\text{million farms}}{\text{year}}$$

0.085000

Relative Change: Suppose P changes from P_0 to P_1

$$\text{Relative change in } P = \frac{\Delta P}{\boxed{P_0}} = \frac{P_1 - P_0}{P_0}$$

Ex: Find the relative change in price of a \$2 increase.

a) gallon of gas

$\$2.40 = P_0$

$\$4.40 = P_1$

$$\frac{4.40 - 2.40}{2.40} = \frac{2.00}{2.40} = 0.833\ldots$$

or $\approx 83\%$

b) cellphone costing \$200.00

$P_0 = \$200.00$

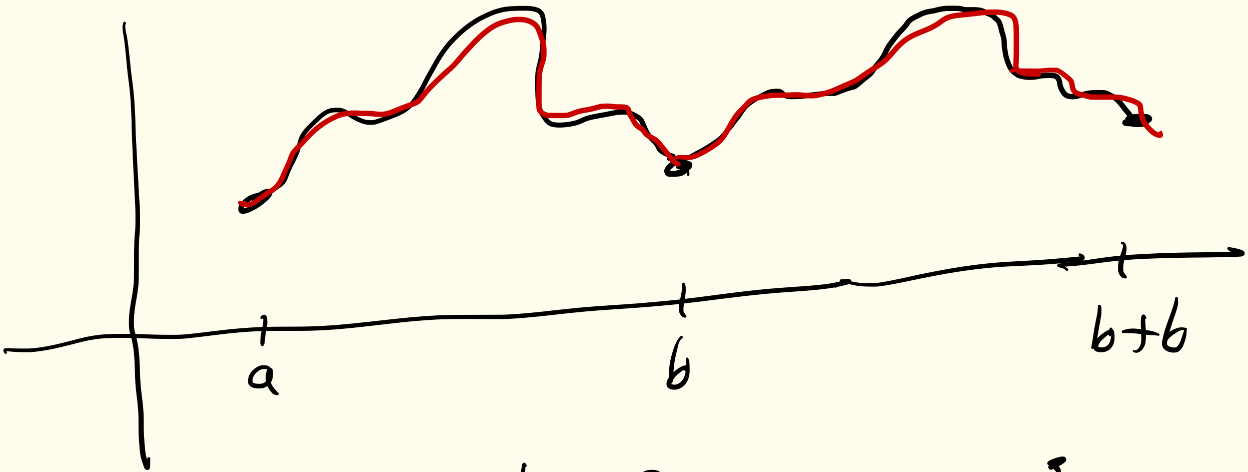
$P_1 = \$200.00 + \$2.00 = 202.00$

$$\frac{202 - 200}{200} = \frac{1}{100} = 0.01$$

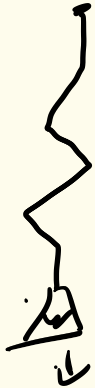
or 1%

1.10: Periodic Functions

(approximation)

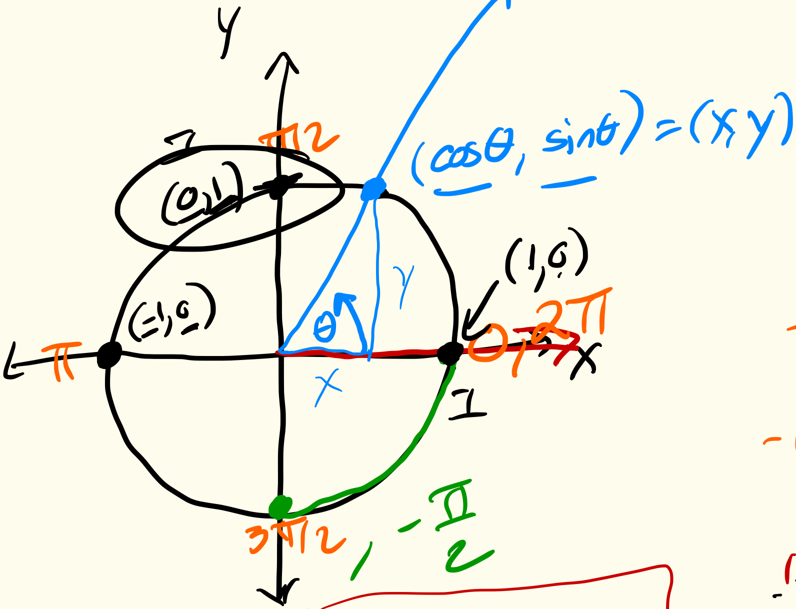


period = $b - a$



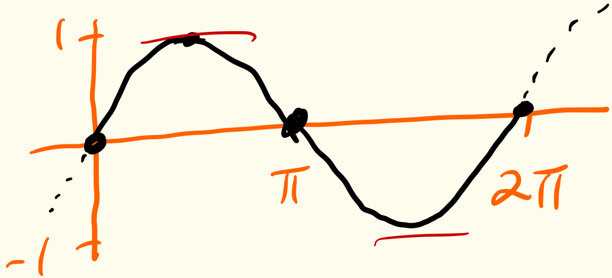
Trigonometric Functions: $\sin(\theta)$ $\cos(\theta)$

θ measured in **RADIANS**

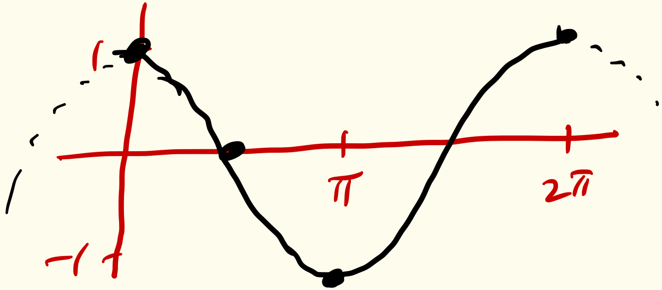


period $= 2\pi$

sin amplitude $= 1$



cos



$$x^2 + y^2 = 1$$

$$\boxed{\begin{array}{c} -1 \le \cos\theta \le 1 \\ -1 \le \sin\theta \le 1 \end{array}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

Family of Curves

$$y = \underline{A}\sin(\underline{B}t) + C \quad \text{or} \quad y = \underline{A}\cos(\underline{B}t) + C$$

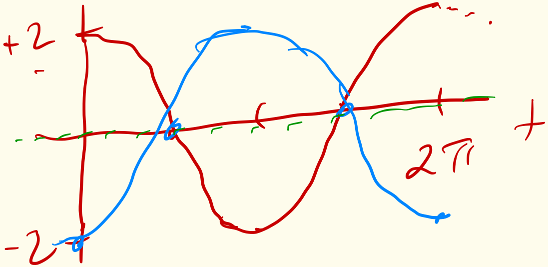
C: vertical shift

vertical shift

$$\boxed{\text{period} = \frac{2\pi}{|B|}}$$

$$\boxed{\text{amplitude} = |A|}$$

Ex: $y = \underline{2}\cos t$



$y = -2\cos t$

$\sin t$: period $2\pi \rightarrow [0, 2\pi]$ or $[-\pi, \pi]$

$\underline{\sin}(2t)$ $\quad 0 \le \underline{2t} \le 2\pi$

$t \in [0, \pi] \Leftarrow \frac{0}{2} \le t \le \frac{2\pi}{2} = \pi$

$t \in [0, \pi] \longrightarrow 2t \in [0, 2\pi]$

period is π

Ex: $y = -\sin\left(\frac{1}{2\pi} t\right)$

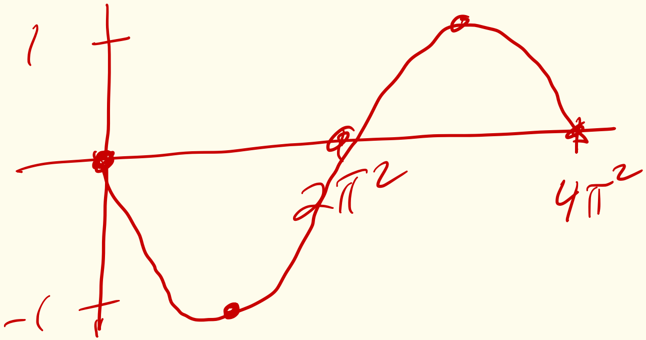
$A = -1$ $C = 0$

$B = \frac{1}{2\pi}$

amplitude $= |A| = \underline{1}$

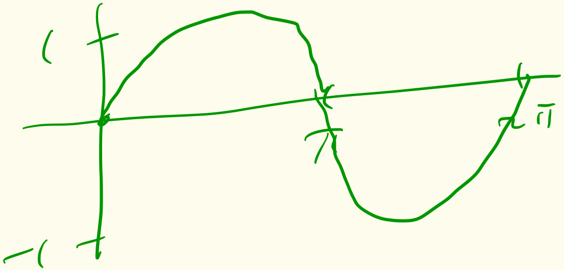
period $= \frac{2\pi}{\left|\frac{1}{2\pi}\right|} = 2\pi \cdot 2\pi$

$= 4\pi^2$

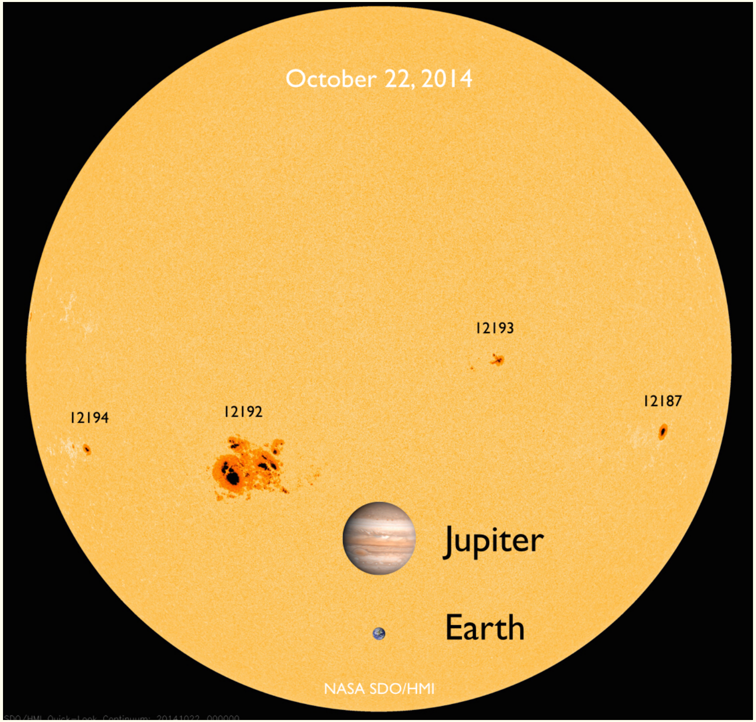


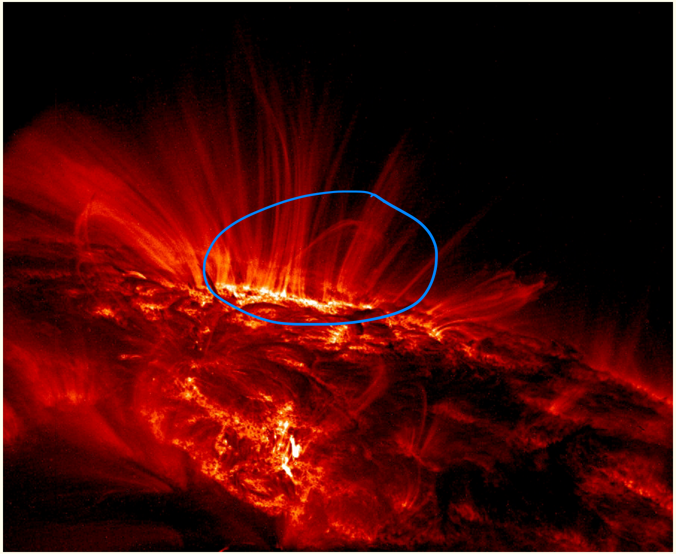
$y = \sin t$

$A = 1,\ B = 1$

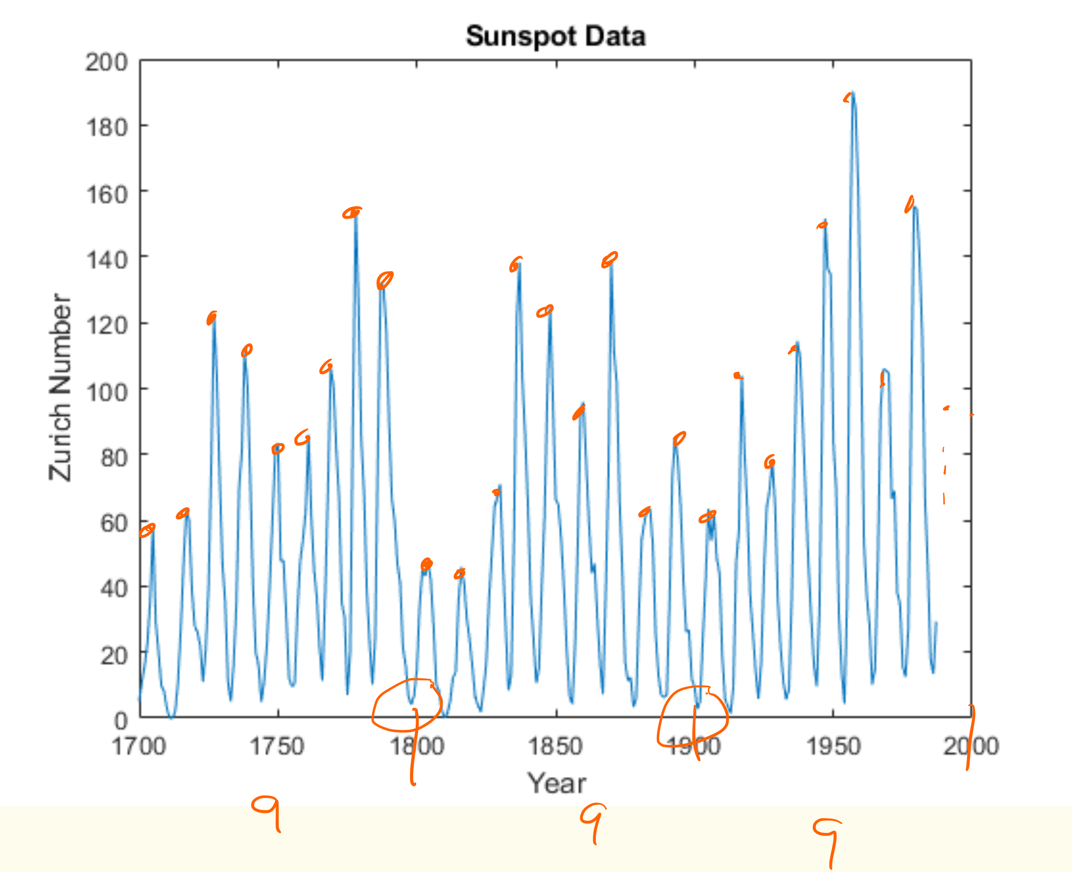


Sunspots

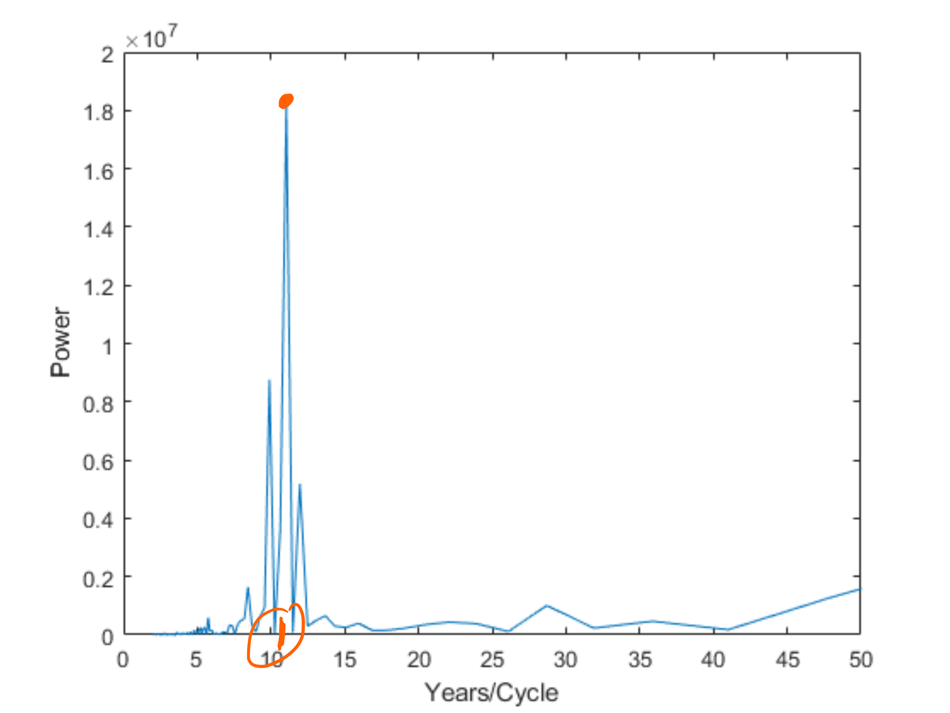

October 22, 2014
12193
12194
12192
12187
Jupiter
Earth
NASA SDO/HMI



Ex: Sunspot Data (www.mathworks.com)



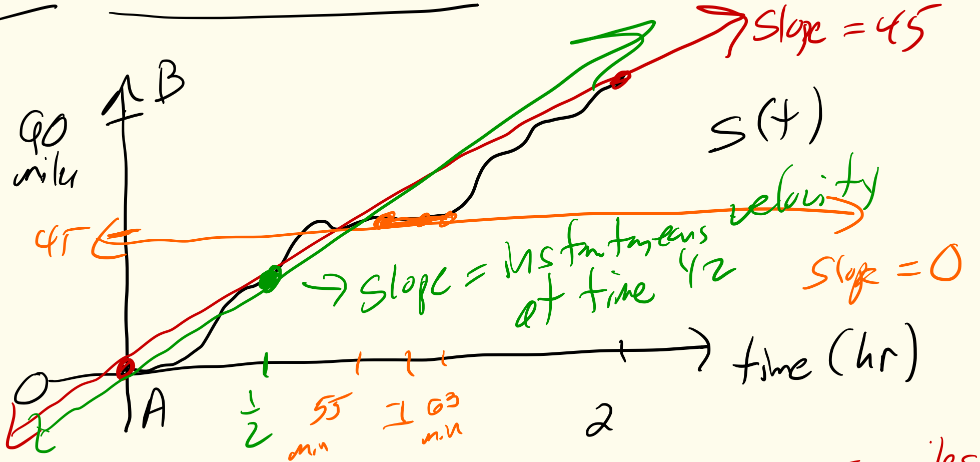
Results :



Period $\approx$ 10 yrs

2.1: Instantaneous Rate of Change (velocity)

$y = s(t)$
position at time t



$$\text{avg velocity trip} = \frac{\Delta s}{\Delta t} = \frac{90-0}{2-0} = 45 \frac{\text{miles}}{\text{hr}} = 45 \text{ mph}$$

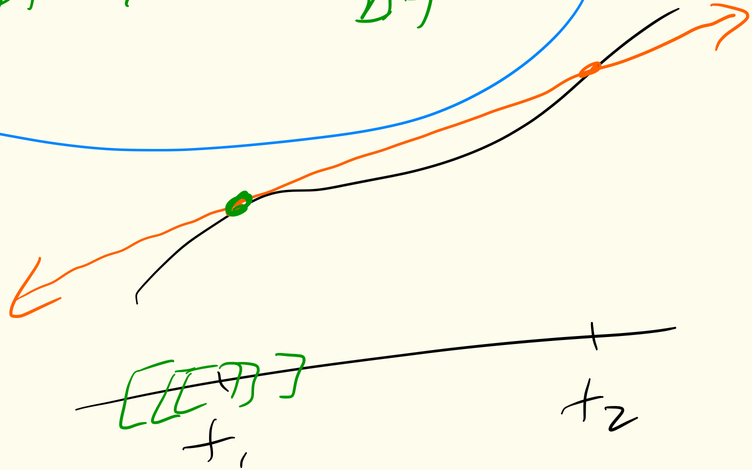
$$\text{avg velocity} = \frac{\Delta s}{\Delta t} = \frac{45-45}{1\frac{3}{60} - \frac{55}{60}} = 0 \text{ mph}$$

instantaneous velocity at 30 min

$[t_1, t_2]$: avg velocity $= \dfrac{S(t_2) - S(t_1)}{t_2 - t_1} = \dfrac{\Delta S}{\Delta t}$

$t_1 \neq t_2$

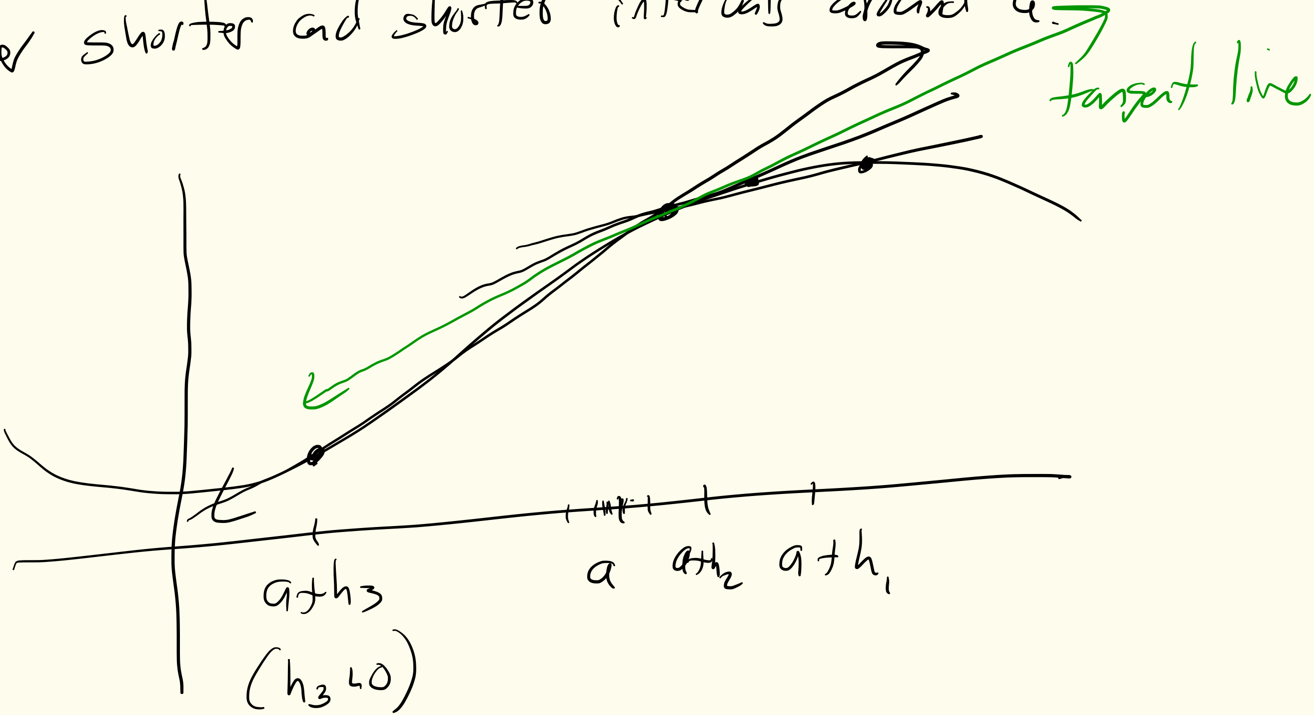
instaneous vel at $t = t_1$: let $\Delta t \to 0$ in $\dfrac{\Delta S}{\Delta t}$



Difficulty

We can't just set $\Delta t = 0$!

Defn: The INSTANTANEOUS RATE OF CHANGE of f at a is the LIMIT of the average rates of change of f over shorter and shorter intervals around a.



Ex: $Q(t) = 25(0.8)^t = 25\left(\frac{4}{5}\right)^t$ (exponential)

is the quantity (mg) in the blood at time t (min).

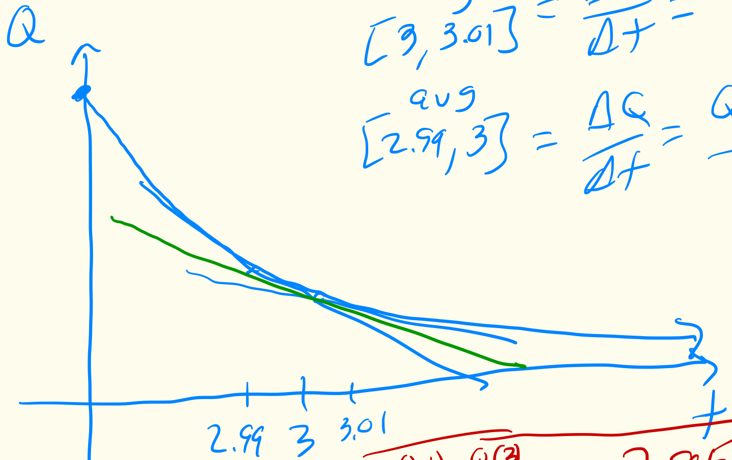
Estimate the instantaneous rate of change of the drug in the body when $t = 3$.

$a = \text{base} = \frac{4}{5} < 1$

$Q(0) = 25 \cdot (4/5)^0 = 25$

$$\underset{[3,\,3.01]}{\text{avg}} = \frac{\Delta Q}{\Delta t} = \frac{Q(3.01) - Q(3)}{3.01 - 3} = -2.8530\ldots \text{ mg/min}$$

$$\underset{[2.99,\,3]}{\text{avg}} = \frac{\Delta Q}{\Delta t} = \frac{Q(3) - Q(2.99)}{3 - 2.99} \approx -2.8594 \text{ mg/min}$$



$\approx$

$$\boxed{\frac{Q(3.01) - Q(3)}{3.01 - 3} = -2.85}$$ NO!

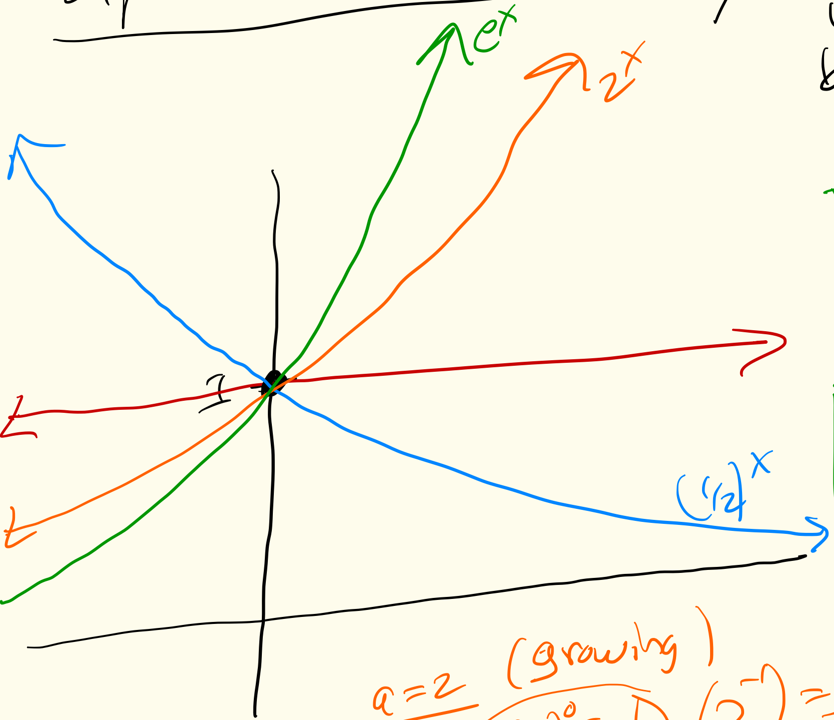
ans ≈ -2.85

Exponential Functions

$y = \underbrace{a}_{\text{base}}^{x}$ $\quad 0 < a < \infty$

$0 < a^x < \infty$

$\boxed{a^0 = 1}$



$y = e^x$

$e \approx 2.7182\ldots$

$\frac{d}{dx}(e^x) = e^x$

$\underline{a = 1}$

$y = 1^x = \underline{1}$

$\underline{a = \frac{1}{2} \text{ (decaying)}}$

$y = \frac{1}{2}^x$

$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$

$\underline{a = 2}$ (growing)

$y = 2^x$

$(2)^0 = 1$

$(2)^1 = 2$

$(2)^2 = 4$

$(2^{-1}) = \frac{1}{2}$