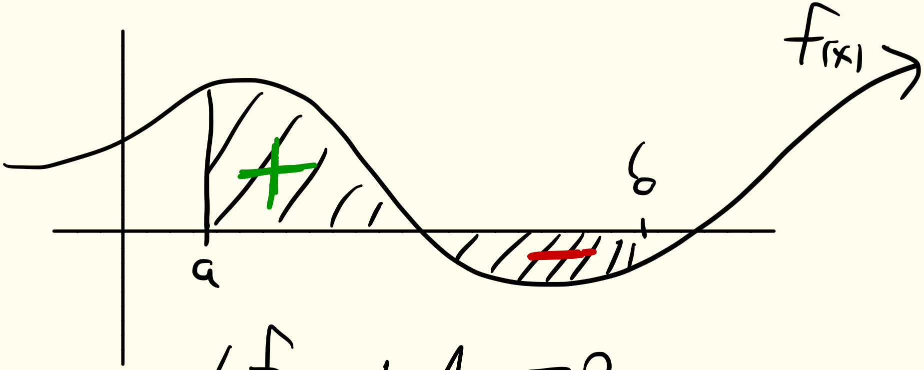
Definite Integral

given: $f(x)$

$[a, b]$



"Area" $= \int_a^b f(x)\,dx \stackrel{\text{defn}}{=}$ LATER

EXACT ANS:

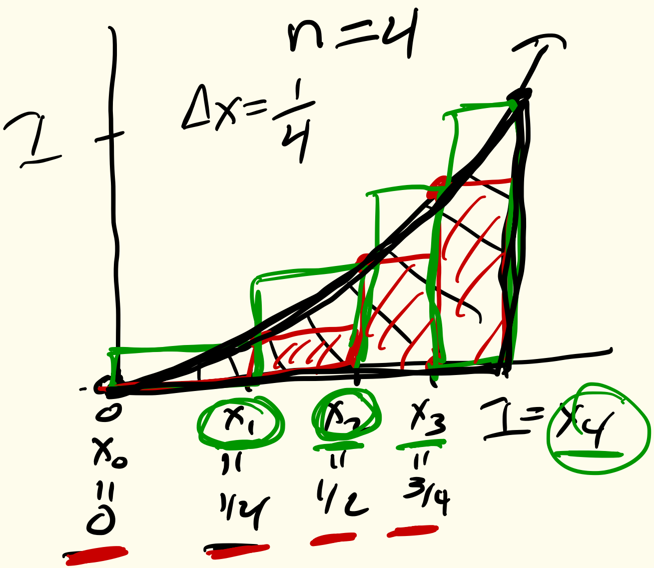
$$\int_a^b f(x)\,dx \stackrel{FTC}{=} F(b)-F(a)$$

where $F'(x) = f(x)$

or $F(x)$ is an ANTI-DERIVATIVE

However, it can be VERY difficult to find $F(x)$!

Ex: $\underbrace{\int_0^1 x^2\,dx}_{\text{Estimate}}$



choose n

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

gridpts: $x_i = a + i\Delta x$

LHS: $[f(x_0) + f(x_1) + f(x_2) + f(x_3)] \cdot \Delta x$

$= [0^2 + \frac{1}{4}^2 + \frac{1}{2}^2 + \frac{3}{4}^2] \cdot \frac{1}{4} = [0 + \frac{1}{16} + \frac{4}{16} + \frac{9}{16}] \cdot \frac{1}{4} = \frac{14}{16} \cdot \frac{1}{4} = \boxed{\frac{7}{32}}$

RHS: $[f(x_1) + f(x_2) + f(x_3) + f(x_4)] \cdot \Delta x$

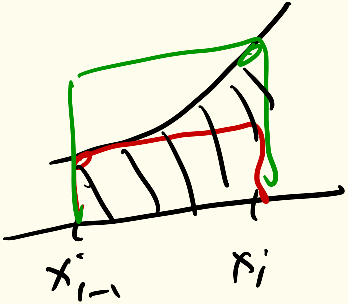
$= [\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}] \cdot \frac{1}{4} = \frac{30}{64} = \boxed{\frac{15}{32}}$

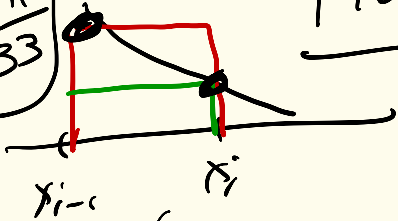
Note: $\frac{7}{32} \le \int_0^1 x^2\,dx \le \frac{15}{32}$ $\boxed{\frac{21}{96} \le \frac{32}{96} \le \frac{45}{96}}$

$\quad\quad \parallel$

$\quad\quad 1/3 = \frac{11}{33}$

Why?





F INCREASING on $[a,b]$: $LHS \le \int_a^b f(x)\,dx \le RHS$

F DECREASING on $[a,b]$: $RHS \le \int_a^b f(x)\,dx \le LHS$

Regardless of $\nearrow$ or $\searrow$

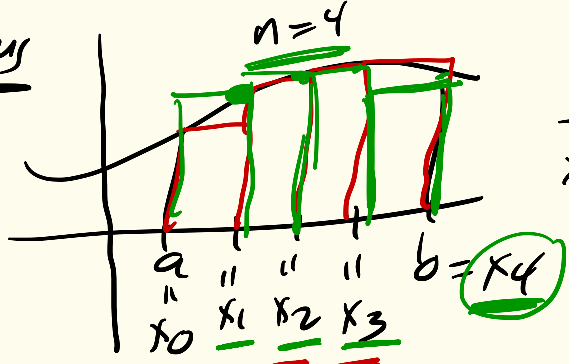
Better Approx: $\frac{LHS + RHS}{2}$

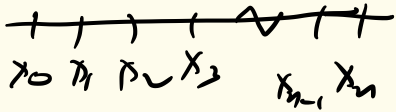
Ex: $\frac{\frac{7}{32} + \frac{15}{32}}{2} = \frac{22}{64} = \boxed{\frac{11}{32}}$

In general: $F(x)$, $[a, b]$ continuous

choose n

$$\Delta x = \frac{b-a}{n}$$





$$LHS = [f(x_0) + f(x_1) + \cdots + f(x_{n-2}) + f(x_{n-1})] \cdot \Delta x$$
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$RHS = [f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)] \cdot \Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

DEFN:

$$\int_a^b f(x)\,dx \overset{\text{defn}}{=} \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(x_i)\Delta x \\ \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x \end{cases} = \begin{cases} \lim\limits_{n\to\infty} LHS \\ \lim\limits_{n\to\infty} RHS \end{cases}$$

Note $\lim\limits_{n\to\infty}$ is the same as $\lim\limits_{\Delta x\to 0}$

Summation Notation

$$\underbrace{0}_{i=0}+\underbrace{1}_{i=1}+\underbrace{2}_{i=2}+\underbrace{3}_{i=3}+\underbrace{4}_{i=4} = \sum_{i=0}^{4} i$$

$$\sum_{i=2}^{6} i^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$\sum_{i=\text{beg}}^{\text{end}} f(i)$$

Gauss

$$\sum_{i=1}^{n} i = 1+2+3+4+\cdots+n$$

$$= \frac{n(n+1)}{2}$$

$\underline{n=2}$: $1+2 = 3$

$\underline{n=5}$: 15

$\underline{n=1000}$: $\frac{1000(1001)}{2}$

$= 500 \cdot 1001$

Ex!

Values for a function $f(t)$ are in the following table. Estimate $\int_{20}^{30} f(t)\,dt$.

$n=5$, $\Delta t = \frac{30-20}{5} = 2$

t	20	22	24	26	28	30
$f(t)$	5	7	11	18	29	45

$LHS = [5+7+11+18+29]\cdot 2 = 140$

$RHS = [7+11+18+29+45]\cdot 2 = 220$

Fi

INCREASING

Better: $\frac{LHS+RHS}{2} = \frac{140+220}{2} = \boxed{180}$