Mimetic Methods
and why they are better

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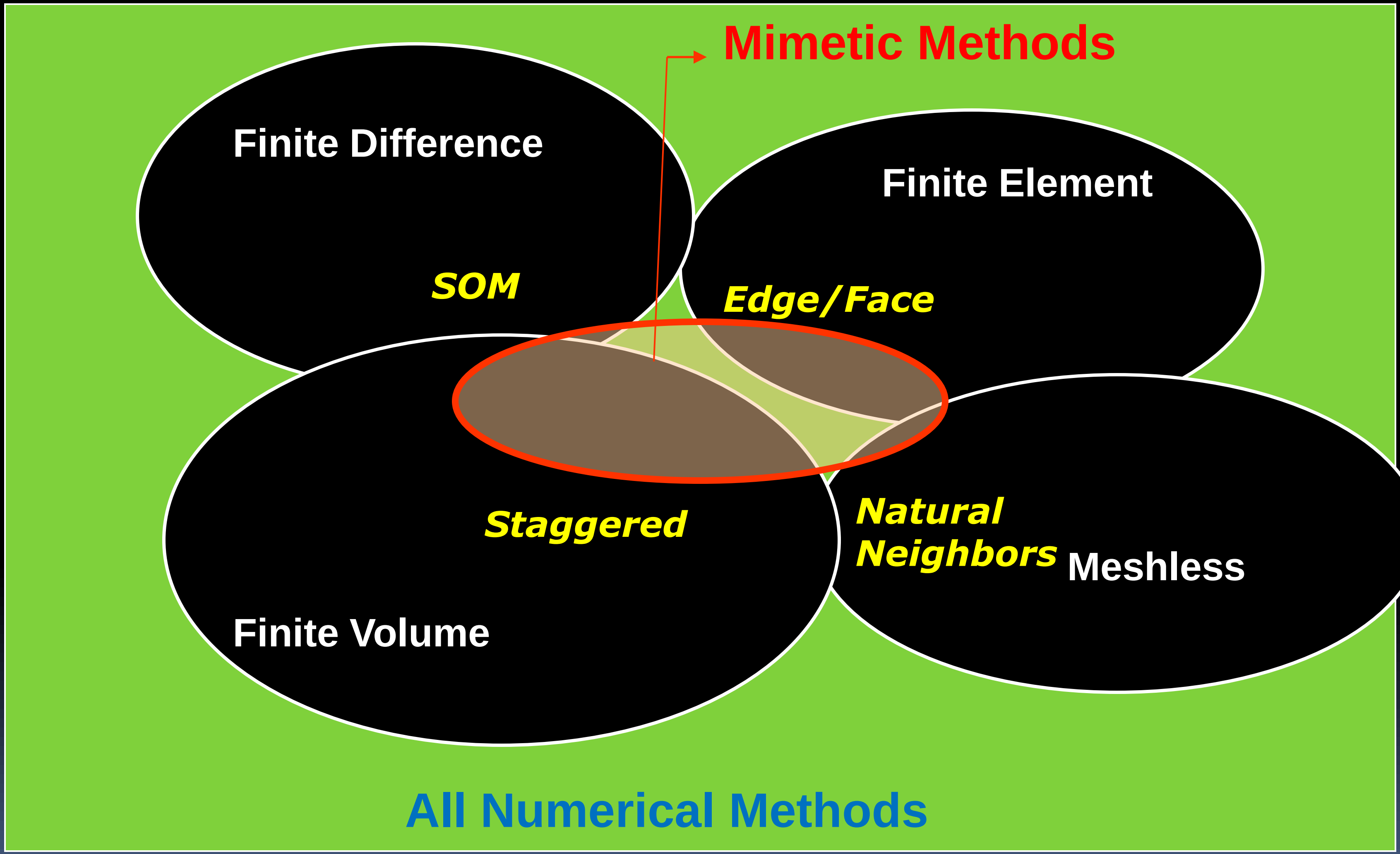
Numerical Methods:
Are more than Mathematics

Math:
- Accuracy
- Stability
- Convergence
- Consistency

Physics:
- Conservation
- Spurious Modes
- Wave propagation
- Maximum/minimum
- Constraints

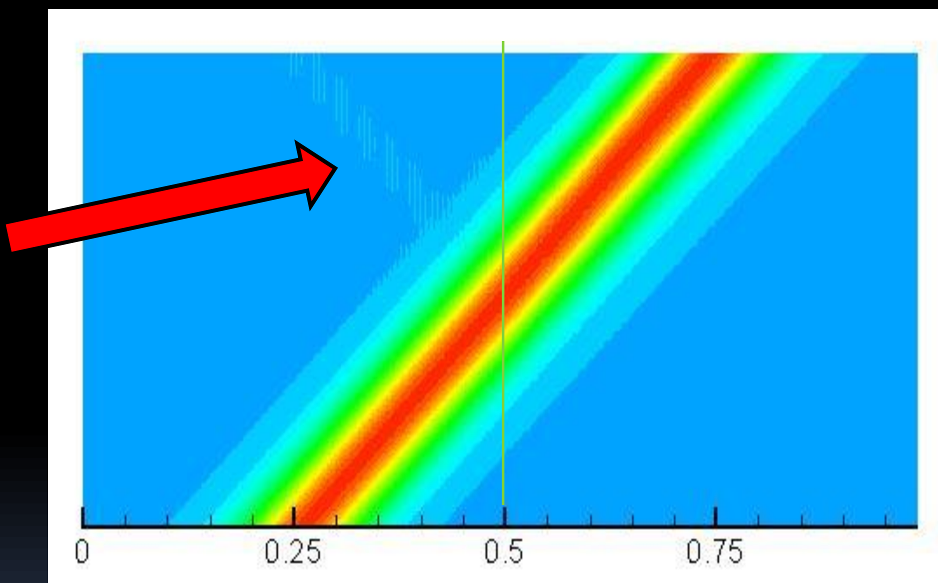
Mimetic methods mimic the physics.

Relationship



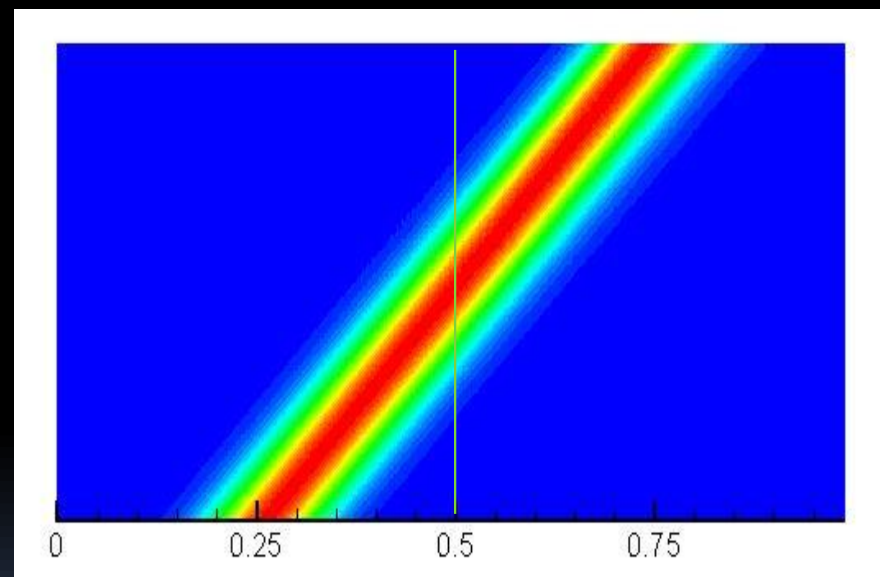
Mimetic Advection $\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = 0$

1D Advection: Change in mesh size (3x more mesh on the right side)



150 **50**

Central



Box Method

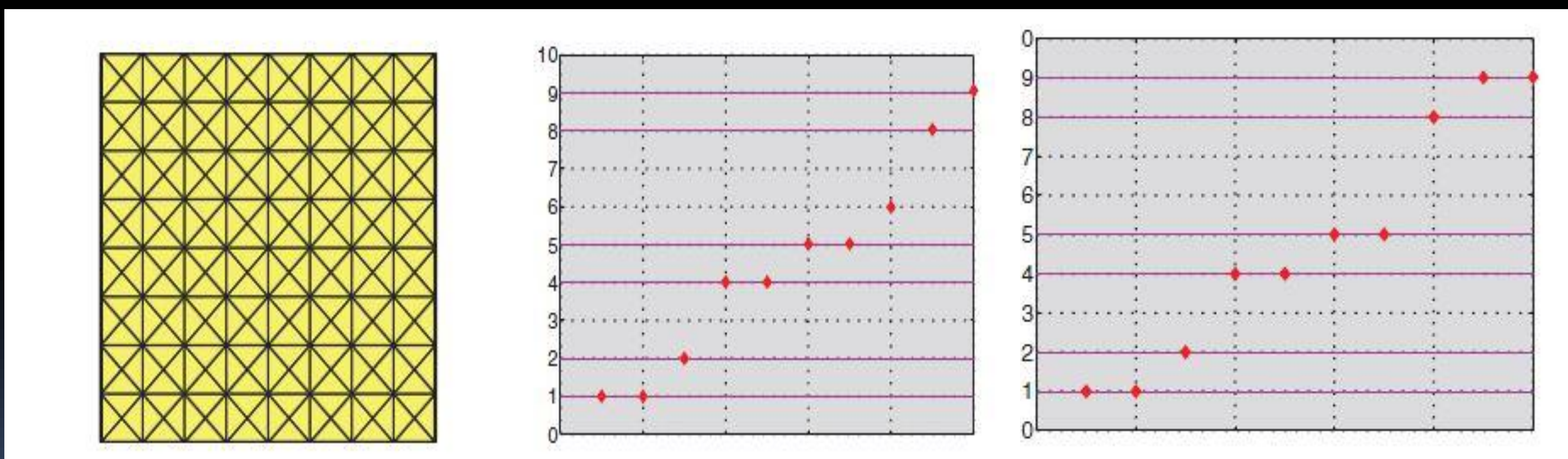
Spurious Wave Reflection

Eigenvalues: Theory

Vector Laplacian

$$\nabla\times\nabla\times\mathbf{v}-\nabla(\nabla\cdot\mathbf{v})=\lambda\mathbf{v} \qquad \lambda=m^2+n^2$$

1,1,2,4,4,5,5,8,..

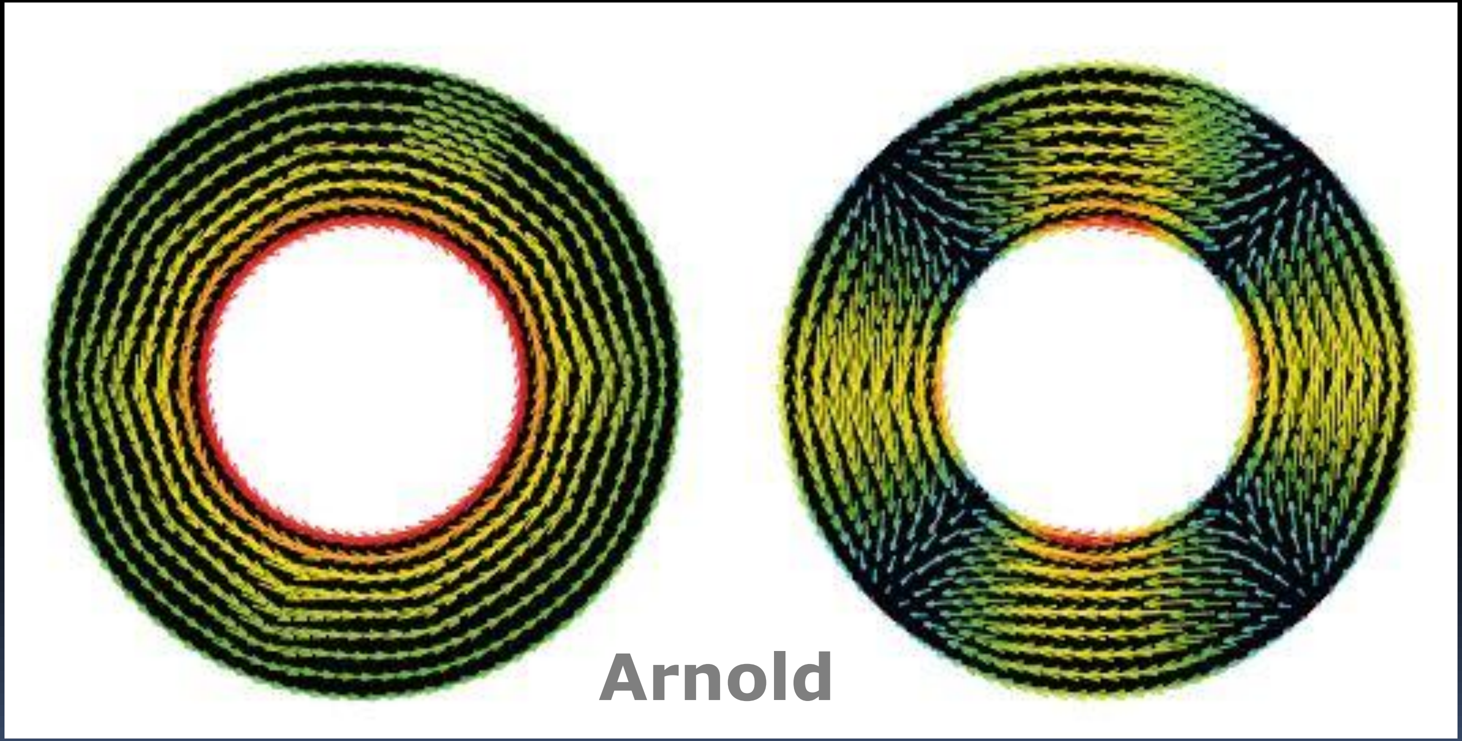


Linear FE **Nedelec FE**

$\mathbf{v}_{node}$ $\qquad u=\mathbf{v}\cdot\mathbf{t}_{edge}$

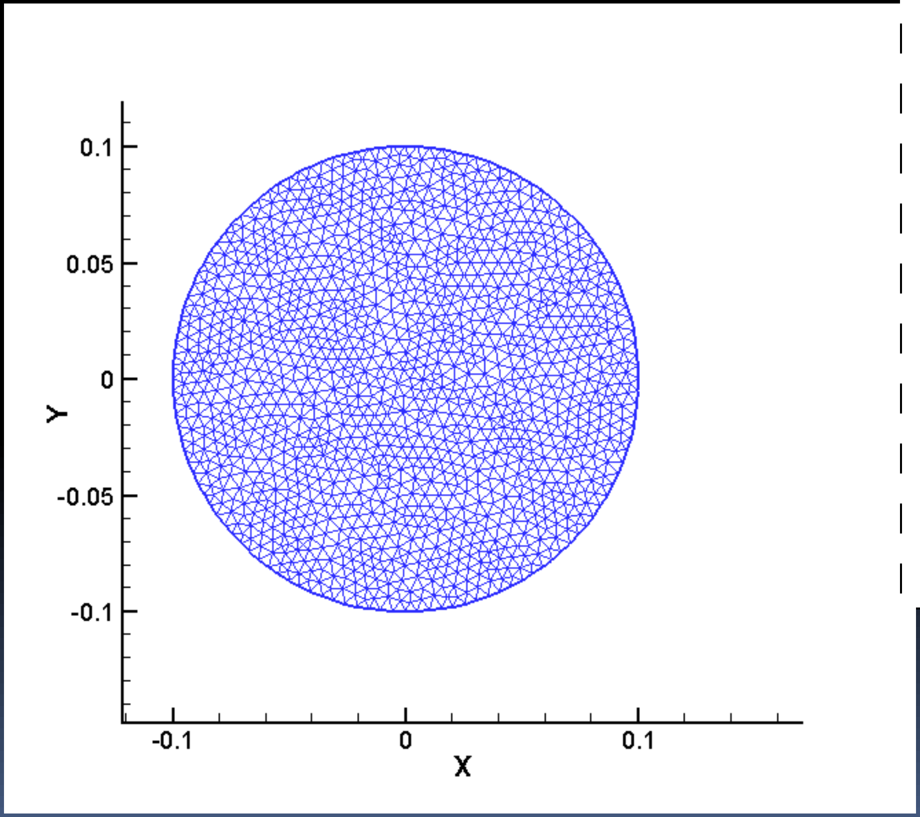
Eigenvectors: Practice

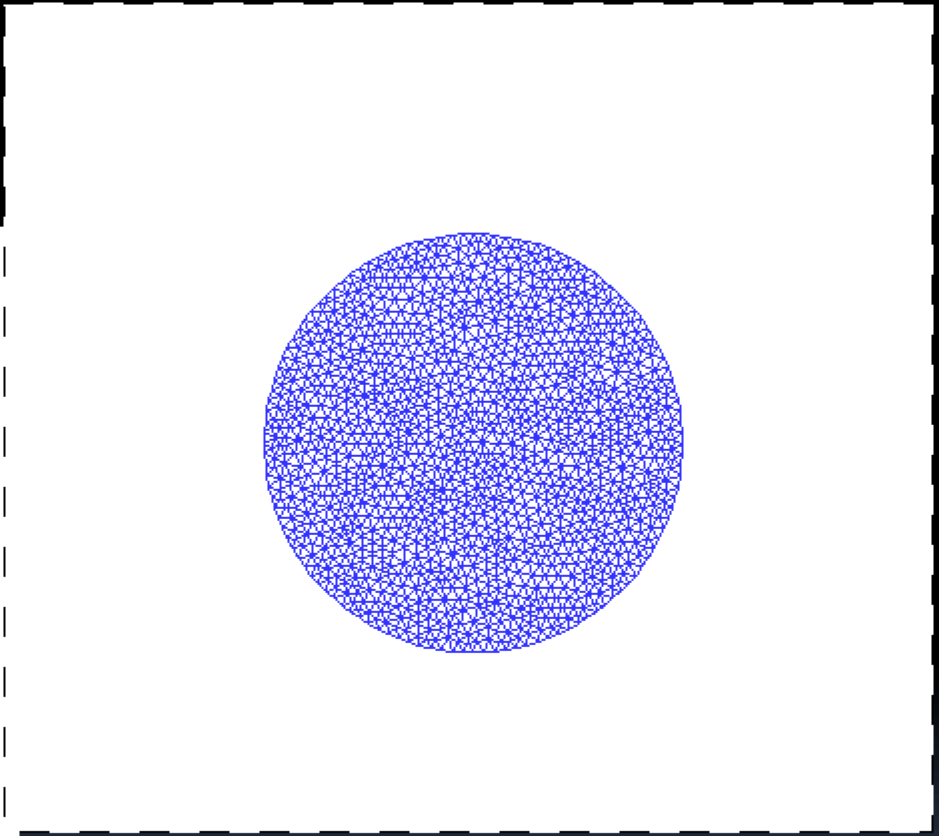
$$\nabla \times \nabla \times \mathbf{v} - \nabla(\nabla \cdot \mathbf{v}) = f$$



Linear FE **Nedelec FE**

Mimetic Surface Tension





Overview

Use Exact Discretization

Separate Discretization	from Approximation	

Separate Discretization
from Approximation

PDE to LA
LA to square LA

- Do ALL discretization exactly.
- This means that the calculus and the physics remain exact.

- Numerical approximation in material laws.
- Which are engineering approximations already.
- Numerical approximation goes with physical approximation.

Discrete Calculus: Part 1

Exact Discretization

$$\frac{\partial a}{\partial t} + \nabla \cdot \mathbf{b} = 0 \quad \Longrightarrow \quad \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \begin{pmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{b}} \end{pmatrix} = \vec{\mathbf{r}}$$

Infinite Dimensional	**Finite Dimensional**
Partial Differential Eqn.	**Matrix Problem**

Basic unknowns are integral quantities.
Collect infinite data into finite groups.

Discrete Calculus: Part 2

Solution <u>requires</u> Approximation

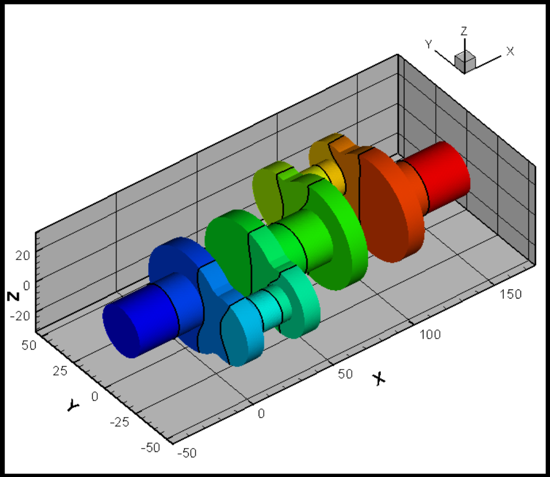
$$\begin{bmatrix} A & B \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \vec{r} \quad \Longrightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} \vec{r} \\ 0 \end{pmatrix}$$

Underdetermined **Unique Square**

Relate discrete unknowns to each other.
This relation is a material law.
Also related to interpolation.
Also related to discrete inner products

Example: Heat Eqn

$$\frac{\partial(\rho c T)}{\partial t} = \nabla \cdot k \nabla T$$



Components of the Physical Equation

$\frac{\partial i}{\partial t} + \nabla \cdot \mathbf{q} = 0$	Conservation of Energy	***Physics***
$\mathbf{g} = \nabla T$	Definition of Gradient	***Math***

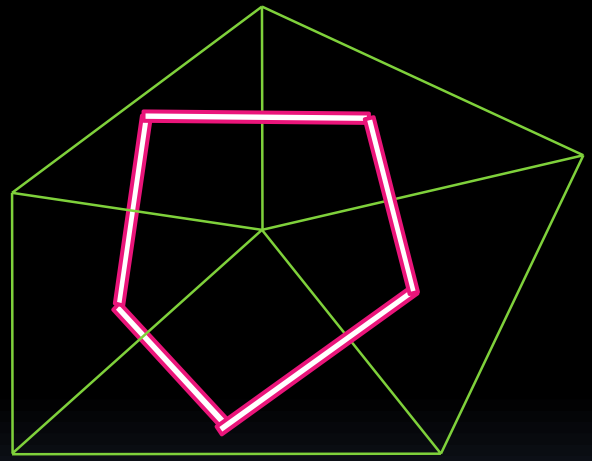
Material Approximation

$\mathbf{q} = -k\mathbf{g}$	Fourier's Law
$i = \rho c T$	Perfectly Caloric Material

Exact Discretization

Perfect representation of Physics and Calculus

- $\int_{\tilde{c}} i dV \,|^{n+1} - \int_{\tilde{c}} i dV \,|^{n} + \sum_{\tilde{f}} \int dt \int_{\tilde{f}} \mathbf{q} \cdot \mathbf{n} dA_{\tilde{f}} = 0$
- $\int_{e} \mathbf{g} \cdot d\mathbf{l} = T_{n2} - T_{n1}$



$$I_{\tilde{c}}^{n+1} - I_{\tilde{c}}^{n} + \mathbf{D} Q_{\tilde{f}} = 0$$

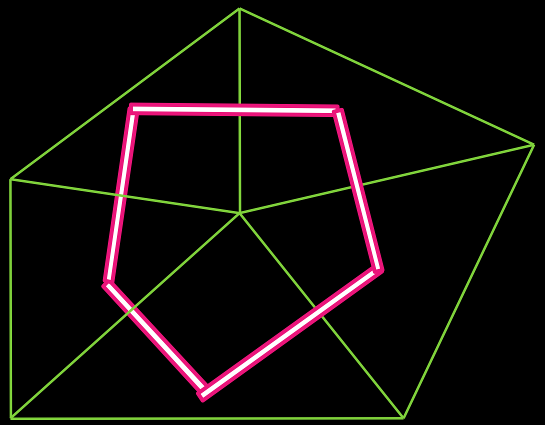
$$g_e = \mathbf{G} T_n$$

Exact

Solution

Numerical Approximation of Constitutive Eqns.

- $$Q_{\tilde{f}} = -M_1 g_e$$
- $$I_{\tilde{c}} = M_2 T_n$$



$$Q_{\tilde{f}} = -k \frac{A_{\tilde{f}}}{L_e} g_e$$

$$I_{\tilde{c}} = \rho c V_{\tilde{c}} T_n$$

Approximate

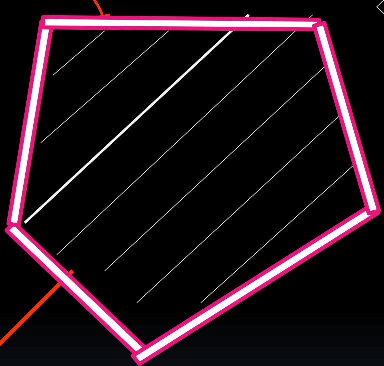
Dependent on Mesh

Dual Mesh Viewpoint

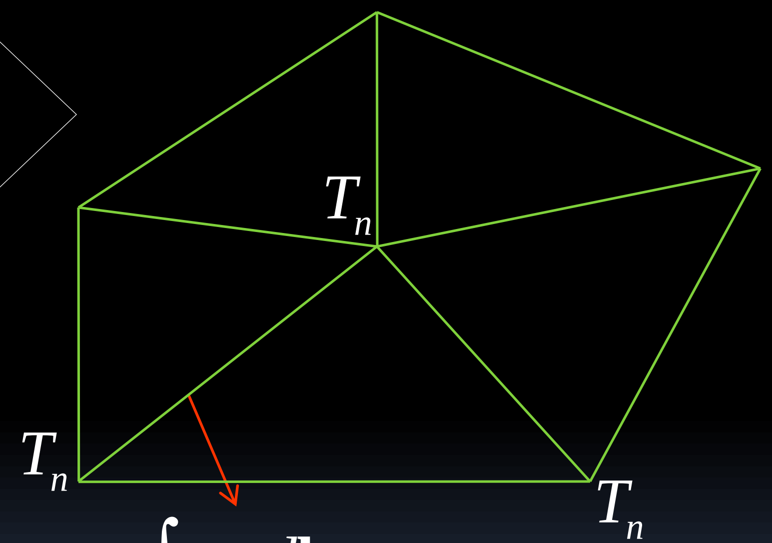
$$I_{\tilde{c}}^{n+1} - I_{\tilde{c}}^{n} + \mathbf{D}Q_{\tilde{f}} = 0$$

$$g_e = \mathbf{G}T_n$$

$$I_{\tilde{c}}^{n+1} = \int_{\tilde{c}} i dV \,|^{n+1}$$



$$I_{\tilde{c}} = \rho c V_{\tilde{c}} T_n$$



$$Q_{\tilde{f}} = \int dt \int_{\tilde{f}} \mathbf{q} \cdot \mathbf{n} dA_{\tilde{f}}$$

$$\int_e \mathbf{g} \cdot d\mathbf{l} = g_e$$

$$Q_{\tilde{f}} = -k \frac{A_{\tilde{f}}}{L_e} g_e$$

Dual Mesh

Primary Mesh

Properties

Conservation of Energy

Entropy Production

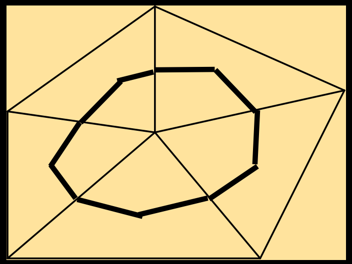
Maximum Principal

Any continuous principle for the PDE …

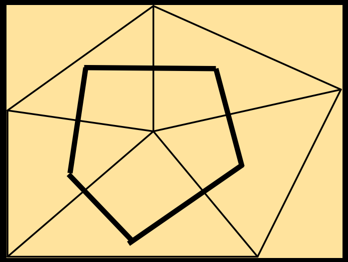
All errors appear as imperfect material properties.

Variations

Choice of the Dual Mesh.



Median Dual



Voronoi Dual

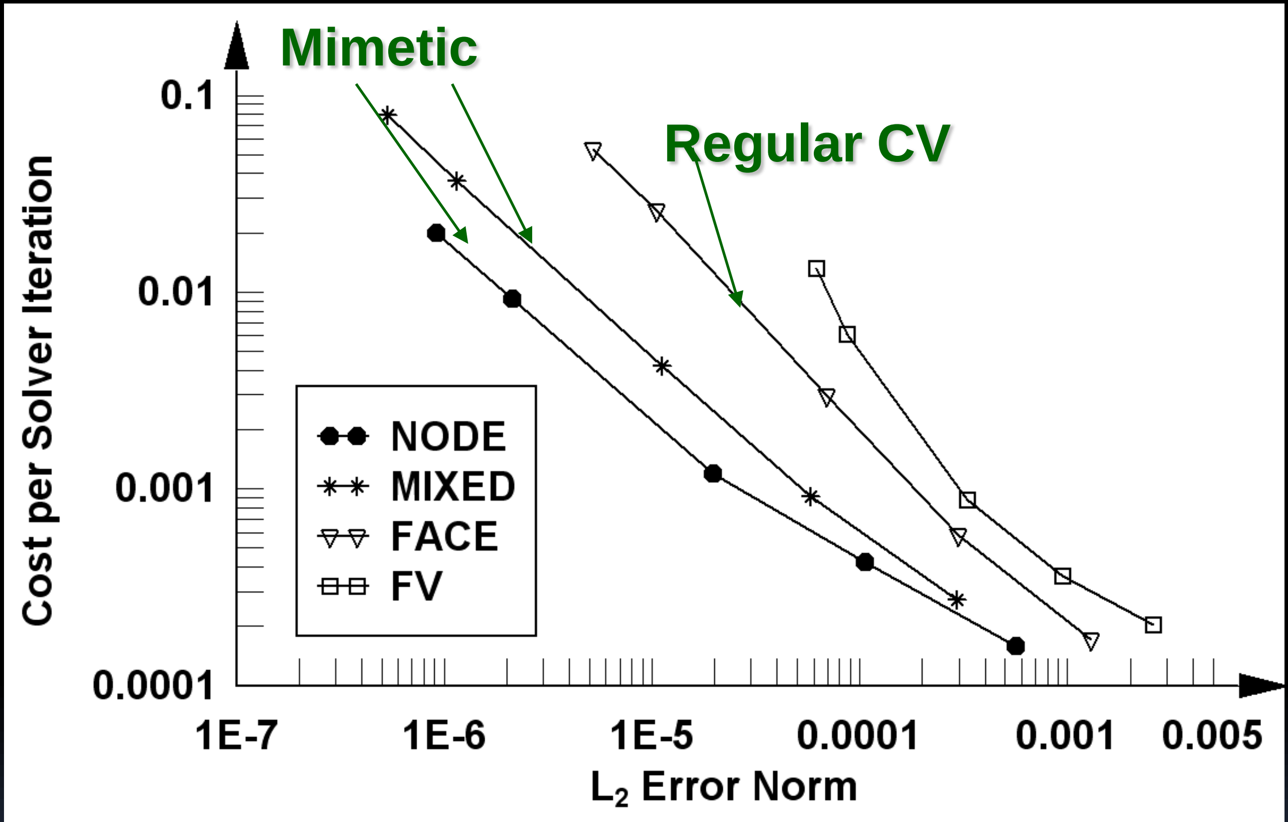
Dual or Primary

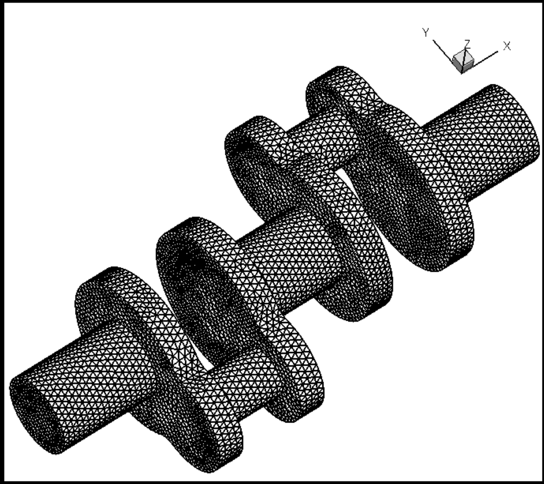
Node centered pressure.
Cell centered pressure.

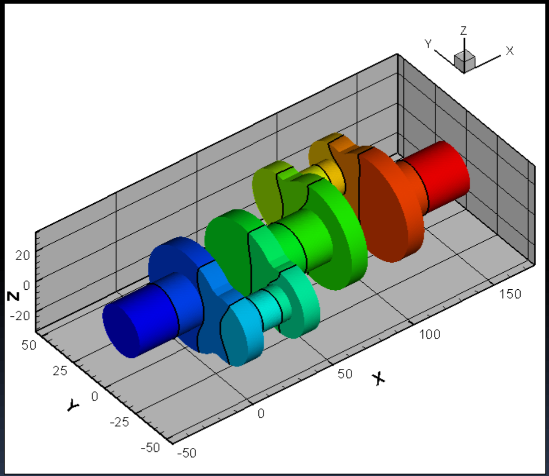
Choice of interpolation.

polynomial reconstruction in cells.
reconstruction in dual cells
weighted interpolations (FE).

Results







- Log scale

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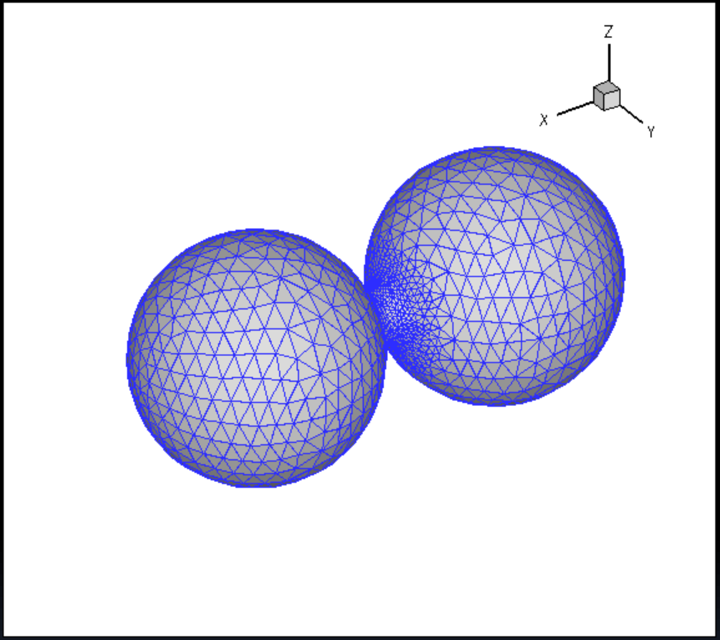
Summary

- Numerical Methods are changing.
- Exact Discretization Approx Solution.
- Works on all types of PDEs



Questions

Navier-Stokes Results