A Variation on the Four Color Problem

Jennifer Li

Department of Mathematics
Louisiana State University
Baton Rouge
(1852) Francis Guthrie:
History

(1852) Francis Guthrie:

“Do four colors suffice to color every planar map so that adjacent countries receive different colors?”
(1852) Francis Guthrie:

“Do four colors suffice to color every planar map so that adjacent countries receive different colors?”

The Four Color Problem or The Map Coloring Problem.
Examples:

- **not a country**
- **not a country**
- **a country**
Example

The planar map below is colored with four colors.
We can depict maps abstractly as graphs.
We can depict maps abstractly as graphs.
We can depict maps abstractly as graphs.
Chromatic Number

Least number of colors needed, denoted by $\chi(G)$
Chromatic Number

Least number of colors needed, denoted by $\chi(G)$
Examples:
Chromatic Number

Least number of colors needed, denoted by $\chi(G)$

Examples:
Another Coloring Problem

- Many attempts to prove the Four Color Problem!
Another Coloring Problem

- Many attempts to prove the Four Color Problem!

1879  A. Kempe
Another Coloring Problem

Many attempts to prove the Four Color Problem!

1879 A. Kempe
1890 P. Heawood
Another Coloring Problem

Many attempts to prove the Four Color Problem!

1879 A. Kempe
1890 P. Heawood

What Heawood did:
Another Coloring Problem

Many attempts to prove the Four Color Problem!

1879 A. Kempe
1890 P. Heawood

What Heawood did:
1. $\times$
Another Coloring Problem

- Many attempts to prove the Four Color Problem!
  
1879 A. Kempe
1890 P. Heawood

- What Heawood did:
  
1. X

2. Red Yellow Green Blue Purple
Another Coloring Problem

- Many attempts to prove the Four Color Problem!

1879 A. Kempe
1890 P. Heawood

- What Heawood did:
  1. X
  2. ● ● ● ● ●
  3. Another question:
Another Coloring Problem

- Many attempts to prove the Four Color Problem!
  - 1879 A. Kempe
  - 1890 P. Heawood

- What Heawood did:
  1. ×
  2. 🟥 🟢 🟡 🟣
  3. Another question:

How many colors are required for graphs embedded on surfaces other than the plane?
Other Surfaces

Sphere = simplest surface
Other Surfaces

Sphere = simplest surface
Sphere + $n$ handles = new surface
Graph Embedding Examples

| Jennifer Li  | (Louisiana State University) | A Variation on the Four Color Problem | May 2, 2015 | 14 / 24 |
$K_5$ is not planar, but is toroidal:
$K_5$ is not planar, but is toroidal:
Faces of a Graph

Example:
Faces of a Graph

Example:
Euler-Poincare Formula

\[ v - e + f = 2 \]

\[ v \equiv \text{vertices} \]
\[ e \equiv \text{edges} \]
\[ f \equiv \text{faces} \]
Euler-Poincare Formula

\( v = \) vertices
\( e = \) edges
\( f = \) faces

For a connected, nonempty planar graph,

\[ v - e + f = 2. \]
Euler-Poincare Formula

\[ v - e + f = 2. \]

The Euler genus \( g \) is defined to be

\[ g = 2 - (v - e + f). \]
Example

plane: $g = 2 - (v - e + f) = 2 - 2 = 0$
Example

plane: \( g = 2 - (v - e + f) = 2 - 2 = 0 \)

\[
\begin{array}{c}
\text{Torus: } g = 2 - (5 - 10 + 5) = 2 - 0 = 2
\end{array}
\]
Example

plane: \( g = 2 - (v - e + f) = 2 - 2 = 0 \)

Torus: \( g = 2 - (5 - 10 + 5) = 2 - 0 = 2 \)
Klein bottle: \( g = 2 - 0 = 2 \)
Example

plane: \( g = 2 - (v - e + f) = 2 - 2 = 0 \)

Torus: \( g = 2 - (5 - 10 + 5) = 2 - 0 = 2 \)

Klein bottle: \( g = 2 - 0 = 2 \)

Projective plane: \( g = 2 - 1 = 1 \)
Heawood’s Conjecture

Heawood used the Euler-Poincare Formula to show:

\[ \chi(G) \leq \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor \]
Heawood’s Conjecture

Heawood used the Euler-Poincare Formula to show:
If $G$ is a loopless graph embedded in surface $S$ with Euler genus $g$, then
Heawood used the Euler-Poincare Formula to show:
If $G$ is a loopless graph embedded in surface $S$ with Euler genus $g$, then

$$\chi(G) \leq \left\lfloor \left( \frac{7 + \sqrt{24g + 1}}{2} \right) \right\rfloor$$
Example

Let $S$ be the torus, which has Euler genus $g = 2$.

By **Heawood’s Formula**: For any loopless graph $G$ embedded in $S$, we have

$$\chi(G) \leq \frac{1}{2} (7 + \sqrt{24 \cdot 2 + 1}) = \frac{1}{2} (7 + 7) = 7$$
Heawood proved the cases where $g > 0$. 
Heawood proved the cases where $g > 0$.

For $g = 0$ (sphere):

$$\chi(G) \leq \frac{1}{2}(7 + \sqrt{24 \cdot 0 + 1}) = \frac{1}{2}(7 + 1) = 4$$

The Four Color Problem!

BUT...
Heawood proved the cases where $g > 0$.

For $g = 0$ (sphere):

$$
\chi(G) \leq \frac{1}{2} (7 + \sqrt{24 \cdot 0 + 1}) = \frac{1}{2} (7 + 1) = 4
$$
Proof of Conjecture

Heawood proved the cases where $g > 0$.

For $g = 0$ (sphere):

$$
\chi(G) \leq \frac{1}{2}(7 + \sqrt{24 \cdot 0 + 1}) = \frac{1}{2}(7 + 1) = 4
$$

The Four Color Problem!
Heawood proved the cases where $g > 0$.

For $g = 0$ (sphere):

$$\chi(G) \leq \frac{1}{2}(7 + \sqrt{24 \cdot 0 + 1}) = \frac{1}{2}(7 + 1) = 4$$

The Four Color Problem!

BUT...
Heawood proved the cases where $g > 0$.

For $g = 0$ (sphere):

$$
\chi(G) \leq \frac{1}{2} (7 + \sqrt{24 \cdot 0 + 1}) = \frac{1}{2} (7 + 1) = 4
$$

The Four Color Problem!

BUT...

Heawood could not prove this case.
Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

colors.
Best Bound?

Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

colors.

Is this the best bound?
Every map on surface $S$ with Euler genus $g$ can be colored with at most
$$\left\lfloor \frac{7 + \sqrt{4g + 1}}{2} \right\rfloor$$
colors.

Is this the best bound? Yes, for every surface EXCEPT

Klein bottle!
Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!
Best Bound?

Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \left( \frac{7 + \sqrt{24g + 1}}{2} \right) \right\rfloor$$

colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!

The formula gives $\chi(G) \leq \left( \frac{1}{2} \right)(7 + \sqrt{24 \cdot 2 + 1}) = \left( \frac{1}{2} \right)(7 + 7) = 7$
Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!

The formula gives $\chi(G) \leq \left(\frac{1}{2}\right)(7 + \sqrt{24 \cdot 2 + 1}) = \left(\frac{1}{2}\right)(7 + 7) = 7$

1934, P. Franklin: Klein bottle requires at most 6 colors.
Every map on surface $S$ with Euler genus $g$ can be colored with at most
\[
\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor
\]
colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!

The formula gives $\chi(G) \leq \left(\frac{1}{2}\right)(7 + \sqrt{24 \cdot 2 + 1}) = \left(\frac{1}{2}\right)(7 + 7) = 7$

1934, P. Franklin: Klein bottle requires at most 6 colors.
1954, G. Ringel: Heawood conjecture gives the best bound for every other surface.
Best Bound?

Every map on surface $S$ with Euler genus $g$ can be colored with at most

$$\left\lfloor \left( \frac{7 + \sqrt{24g + 1}}{2} \right) \right\rfloor$$

colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!

The formula gives $\chi(G) \leq \left( \frac{1}{2} \right)(7 + \sqrt{24 \cdot 2 + 1}) = \left( \frac{1}{2} \right)(7 + 7) = 7$

1934, P. Franklin: Klein bottle requires at most 6 colors.

1954, G. Ringel: Heawood conjecture gives the best bound for every other surface.

Every map on surface $S$ with Euler genus $g$ can be colored with at most
\[
\left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor
\]
colors.

Is this the best bound? Yes, for every surface EXCEPT Klein bottle!

The formula gives $\chi(G) \leq \left(\frac{1}{2}\right)(7 + \sqrt{24 \cdot 2 + 1}) = \left(\frac{1}{2}\right)(7 + 7) = 7$

1934, P. Franklin: Klein bottle requires at most 6 colors.
1954, G. Ringel: Heawood conjecture gives the best bound for every other surface.
1974, Ringel-Youngs Theorem published.
A Correct Solution!
1977, Appel, Haken: The Four Color Problem
1977, Appel, Haken: The Four Color Problem
The Four Color Problem: unsolved for nearly 100 years!
The Four Color Problem: unsolved for nearly 100 years!

Moreover,
The Four Color Problem: unsolved for nearly 100 years!

Moreover,
- Four Color Problem:

Heawood's conjecture: application of the quadratic formula.
The Four Color Problem: unsolved for nearly 100 years!

Moreover,

- Four Color Problem: 1200 hours of computer time on University of Illinois supercomputer.
The Four Color Problem: unsolved for nearly 100 years!

Moreover,

- Four Color Problem: 1200 hours of computer time on University of Illinois supercomputer.
- Heawood’s conjecture:
The Four Color Problem: unsolved for nearly 100 years!

Moreover,
- Four Color Problem: \textit{1200 hours of computer time} on University of Illinois supercomputer.
- Heawood’s conjecture: application of \textit{the quadratic formula}. 
Thanks to:

- My project advisor, Professor Oporowski!
- Federico Salmoiraghi!