DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 233 **SOME SOLUTIONS TO EXAM 2** **Fall 2018**

“Version A” refers to the regular exam and ”Version B” to the make-up

1. Version A. A particle moves along the curve $\mathbf{r}(t) = (t^3/3, t^2, 2t)$. Find the length of the path traveled by the particle between $t = 1$ and $t = 3$.

   Solution.

$$\int_1^3 \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} dt = \int_1^3 \sqrt{(t^2)^2 + (2t)^2 + (2)^2} dt = \int_1^3 \sqrt{t^4 + 4t^2 + 4} dt$$

$$= \int_1^3 \sqrt{(t^2+2)^2} dt = \int_1^3 (t^2 + 2) dt = \frac{3^3}{3} + 2 * 3 - \frac{1^3}{3} - 2 * 1 = 12\frac{2}{3}$$

   Version B. A particle moves along the curve $\mathbf{r}(t) = (t^4/4, (\sqrt{6}/3)t^3, (3/2)t^2)$. Find the length of the path traveled by the particle between $t = 1$ and $t = 2$.

   Solution.

$$\int_1^2 \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} dt = \int_1^2 \sqrt{(t^3)^2 + (\sqrt{6}t^2)^2 + (3t)^2} dt =$$

$$= \int_1^2 \sqrt{(t^3+3t)^2} dt = \int_1^2 (t^3 + 3t) dt = \frac{2^4}{4} + 3 * \frac{2^2}{2} - \frac{1^4}{4} - 3 * \frac{1^2}{2} = 8\frac{1}{4}$$

2. Version A. Find the position vector function of a particle that has an acceleration function

$$\mathbf{a}(t) = e^{t/2}\mathbf{i} + \mathbf{k},$$

   an initial velocity $\mathbf{v}(0) = 2\mathbf{j}$, and an initial position $\mathbf{r}(0) = 0$.

   Solution.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = 2e^{t/2}\mathbf{i} + t\mathbf{k} + \mathbf{C}$$

Since

$$\mathbf{v}(0) = 2\mathbf{i} + \mathbf{C} = 2\mathbf{j},$$

we have $\mathbf{C} = 2\mathbf{j} - 2\mathbf{i}$, and so

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = (2e^{t/2} - 2)\mathbf{i} + 2\mathbf{j} + t\mathbf{k}.$$

Integrating once again,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = (4e^{t/2} - 2t)\mathbf{i} + 2t\mathbf{j} + t^2/2\mathbf{k} + \mathbf{C}'$$

Since $\mathbf{r}(0) = 0 = 4\mathbf{i}\mathbf{C}'$, we have $\mathbf{C}' = -4\mathbf{i}$. Finally,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = (4e^{t/2} - 2t - 4)\mathbf{i} + 2t\mathbf{j} + t^2/2\mathbf{k}.$$

Version B. Find the position vector function of a particle that has an acceleration function

$$\mathbf{a}(t) = \cos(t/2)\mathbf{i} + \mathbf{k},$$

an initial velocity $\mathbf{v}(0) = 3\mathbf{j}$, and an initial position $\mathbf{r}(0) = 0$.

Solution.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = 2\sin(t/2)\mathbf{i} + t\mathbf{k} + \mathbf{C}$$

Since

$$\mathbf{v}(0) = \mathbf{C} = 3\mathbf{j},$$

we have $\mathbf{C} = 3\mathbf{j}$, and so

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = (2\sin(t/2))\mathbf{i} + 3\mathbf{j} + t\mathbf{k}.$$

Integrating once again,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = -4\cos(t/2)\mathbf{i} + 3t\mathbf{j} + t^2/2\mathbf{k} + \mathbf{C}'$$

Since $\mathbf{r}(0) = 0 = -4\mathbf{i} + \mathbf{C}'$, we have $\mathbf{C}' = 4\mathbf{i}$.

Finally,

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = (-4\cos(t/2) + 4)\mathbf{i} + 3t\mathbf{j} + t^2/2\mathbf{k}.$$

3. Version A. Consider the function

$$f(x, y) = \frac{x^3 y}{x^6 + y^2}.$$

(a) Find the limit of $f(x, y)$ as (x, y) approaches the origin *along a straight line* of slope m, where m is a real number.

(b) Find the limit of $f(x, y)$ as (x, y) approaches the origin *along the curve* $y = x^3$.

(c) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain why or why not.

Solution. (a) For $x \neq 0$,

$$f(x,mx) = \frac{x^3(mx)}{x^6+(mx)^2} = \frac{mx^2}{x^4+m^2}.$$

The right side is as continuous function of x defined everywhere.

$$\lim_{x\to 0} \frac{mx^2}{x^4+m^2} = \frac{m0^2}{0^4+m^2} = 0.$$

(b) For $x \neq 0$,

$$f(x,x^3) = \frac{x^3(x^3)}{x^6+(x^3)^2} = \frac{1}{2}.$$

$$\lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

(c) The limit does not exist at the origin because we found two different paths approaching the origin for which the function approached two different numbers.

Version B. Consider the function

$$f(x,y) = \frac{x^2y}{x^4+2y^2}.$$

(a) Find the limit of the $f(x,y)$ as (x,y) approaches the origin *along a straight line* of slope m, where m is a real number.

(b) Find the limit of the $f(x,y)$ as (x,y) approaches the origin *along the curve* $y = x^2$.

(c) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain why or why not.

Solution.

(a) For $x \neq 0$,

$$f(x,mx) = \frac{x^2(mx)}{x^4+2(mx)^2} = \frac{mx}{x^2+2m^2}.$$

The right side is as continuous function of x defined everywhere.

$$\lim_{x\to 0} \frac{mx}{x^2+2m^2} = \frac{m(0)}{0^2+m^2} = 0.$$

(b) For $x \neq 0$,

$$f(x,x^2) = \frac{x^2(x^2)}{x^4+2(x^2)^2} = \frac{1}{3}.$$

$$\lim_{x\to 0} \frac{1}{3} = \frac{1}{3}.$$

(c) The limit does not exist at the origin because we found two different paths approaching the origin for which the function approached two different numbers.

Solution.

4. Version A.

(a) Find the linear approximation to $f(x, y) = \sqrt{xy+1}$ at the point $(4, 6)$, and use this to estimate $f(3.9, 5.9)$.
(b) The volume of a square pyramid is measured as 270 cubic centimeters with a possible error of ± 3 cubic centimeters. Its height is measured as 10 centimeters, with a possible error of ± 0.1 centimeters. Use differentials to estimate the maximum error in calculating the side length of the square base from the measured volume and height. You must include units in your final answer. (Recall that the volume of a pyramid with a square base of side length l and height h is $V = \frac{1}{3}l^2 h$.)

Solution. (a) The linear approximation to $f(x, y)$ at (x_0, y_0) is given by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,.$$

Thus, we compute

$$f(4, 6) = \sqrt{(4)(6) + 1} = \sqrt{24 + 1} = \sqrt{25} = 5\,,$$

$$f_x(x, y) = \frac{y}{2\sqrt{xy+1}} \implies f(4, 6) = \frac{3}{5}\,,$$

$$f_y(x, y) = \frac{x}{2\sqrt{xy+1}} \implies f(4, 6) = \frac{2}{5}\,,$$

and so

$$L(x, y) = 5 + \frac{3}{5}(x - 4) + \frac{2}{5}(x - 6)\,.$$

From this, we get the approximation

$$f(3.9, 5.9) \approx L(3.9, 5.9) = 5 + \frac{3}{5}(-0.1) + \frac{2}{5}(-0.1) = 5 - 0.1 = 4.9\,.$$

(b) The base side length, in terms of V and h, is

$$l = \sqrt{\frac{3V}{h}} = \sqrt{\frac{3(270\,\text{cm}^3)}{10\,\text{cm}}} = 9\,\text{cm}\,.$$

The differential is then

$$dl = \frac{1}{2}\sqrt{\frac{3}{Vh}}\,dV - \frac{1}{2}\sqrt{\frac{3V}{h^3}}\,dh = \frac{3}{2lh}\,dV - \frac{3V}{2lh^2}\,dh = \frac{3}{2lh}\,dV - \frac{l}{2h}\,dh\,.$$

Note that one can get the second expression for the differential dl by differentiating $l^2 = 3V/h$, and one can also compute the total differential for V and solve for dl to obtain the third, equivalent expression. Evaluating the differential for the given measurements gives

$$dl = \frac{1}{60\,\text{cm}^2}\,dV - \frac{9}{20}\,dh$$

Plugging in errors of $\pm 3\,\text{cm}^3$ for dV and $\mp 0.1\,\text{cm}$ for dh will give an estimate of maximum error in the measurement of l as

$$dl = \pm\frac{1}{20}\,\text{cm} \pm \frac{9}{200}\,\text{cm} = \pm\frac{19}{200}\,\text{cm}\,.$$

Version B. (a) Explain why $f(x,y)=\sqrt{x^2+y^2+1}$ is differentiable at $(4,8)$, and then find the equation of the tangent plane to graph of $z=f(x,y)$ when $x=4$ and $y=8$.
(b) The volume of a square pyramid is measured as 30 cubic centimeters with a possible error of ± 1 cubic centimeters. Its height is measured as 10 centimeters, with a possible error of ± 0.3 centimeters. Use differentials to estimate the maximum error in calculating the side length of the square base from the measured volume and height. You must include units in your final answer. (Recall that the volume of a pyramid with a square base of side length l and height h is $V=\frac{1}{3}l^2h$.)

Solution. (a) The partials of f are

$$f_x(x,y)=\frac{x}{\sqrt{x^2+y^2+1}}\,,\qquad f_y(x,y)=\frac{y}{\sqrt{x^2+y^2+1}}\,.$$

These functions are algebraic functions of x and y and thus are continuous on their domains, and so in particular are continuous at $(4,8)$. Since f has continuous first partials at $(4,8)$, f is differentiable there.

The tangent plane is then given by

$$z-f(4,8)=f_x(4,8)(x-4)+f_y(4,8)(y-8)$$

which gives

$$z-9=\frac{4}{9}(x-4)+\frac{8}{9}(y-8)\,,$$

which may be simplified to

$$z=\frac{4x+8y+1}{9}\,.$$

(b) The base side length, in terms of V and h, is

$$l=\sqrt{\frac{3V}{h}}=\sqrt{\frac{3(30\,\mathrm{cm}^3)}{10\,\mathrm{cm}}}=3\,\mathrm{cm}\,.$$

The differential is then

$$dl=\frac{1}{2}\sqrt{\frac{3}{Vh}}\,dV-\frac{1}{2}\sqrt{\frac{3V}{h^3}}\,dh=\frac{3}{2lh}\,dV-\frac{3V}{2lh^2}\,dh=\frac{3}{2lh}\,dV-\frac{l}{2h}\,dh\,.$$

Note that one can get the second expression for the differential dl by differentiating $l^2=3V/h$, and one can also compute the total differential for V and solve for dl to obtain the third, equivalent expression. Evaluating the differential for the given measurements gives

$$dl=\frac{1}{20\,\mathrm{cm}^2}\,dV-\frac{3}{20}\,dh$$

Plugging in errors of $\pm 1\,\mathrm{cm}^3$ for dV and $\mp 0.3\,\mathrm{cm}$ for dh will give an estimate of maximum error in the measurement of l as

$$dl=\pm\frac{1}{20}\,\mathrm{cm}\pm\frac{9}{200}\,\mathrm{cm}=\pm\frac{19}{200}\,\mathrm{cm}\,.$$

5. Version A. (a) Find the directional derivative of the function $f(x,y) = \ln(x^2 + y^2)$ in the direction of $\mathbf{v} = \langle 3, -2 \rangle$ at the point $P = (1, 0)$.
(b) Find the maximum possible directional derivative of the function $f(x,y) = \ln(x^2 + y^2)$ at the point $P = (1, 0)$ and the unit vector in the direction in which it occurs.
Solution.

$$\text{(a) } \nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

so at $P = (1, 0)$, we have $\nabla f(1, 0) = \langle 2, 0 \rangle$. The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \right\rangle.$$

Then $D_{\mathbf{u}} f(1,0) = \nabla f(1,0) \cdot \mathbf{u} = \dfrac{6}{\sqrt{13}}$

(b) The maximum possible directional derivative is equal to the magnitude of the gradient vector and occurs in its direction. Thus the maximum directional derivative is equal to $||\nabla f(1,0)|| = 2$ and the unit vector in which it occurs is equal to $\nabla f(1,0)/||\nabla f(1,0)|| = \langle 1, 0 \rangle$.

Version B. (a) Find the directional derivative of the function $f(x,y) = xe^{-y}$ in the direction of $\mathbf{v} = \langle 1, 2 \rangle$ at the point $P = (1, 0)$.
(b) Find the maximum possible directional derivative of the function $f(x,y) = xe^{-y}$ at the point $P = (1, 0)$ and the unit vector in the direction in which it occurs.

Solution.

(a)

$$\nabla f = \langle e^{-y}, -xe^{-y} \rangle$$

so at $P = (1, 0)$, we have $\nabla f(1, 0) = \langle 1, -1 \rangle$. The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

Then $D_{\mathbf{u}} f(1,0) = \nabla f(1,0) \cdot \mathbf{u} = -\frac{1}{\sqrt{5}}$.

(b) The maximum possible directional derivative is equal to the magnitude of the gradient vector and occurs in its direction. Thus the maximum directional derivative is equal to $||\nabla f(1,0)|| = \sqrt{2}$ and the unit vector in which it occurs is equal to $\nabla f(1,0)/||\nabla f(1,0)|| = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$.

6. Version A. Find all local maxima, minima, and saddle points of $f(x,y) = x^3 - 3x + x^2y^2$. Be sure to specify the type of each point you find.

Solution. First we find all the critical points of the function f. Since f has continuous partial derivatives, we have that if a critical point occurs at a point (x, y), then

$$\nabla f(x,y) = \langle\ f_x(x,y), f_y(x,y) \rangle = \langle 3x^2 - 3 + 2xy^2, 2x^2y \rangle = \langle 0, 0 \rangle$$

We have

$$2x^2y = 0 \Longrightarrow x = 0 \text{ or } y = 0$$

if $x = 0$ then by $3x^2 - 3 + 2xy^2 = 0$ we have $3 = 0$ which is a contradiction.
if $y = 0$ then by $3x^2 - 3 + 2xy^2 = 0$ we have $3x^2 - 3 = 0$ and therefore $x = \pm 1$.
therefore the only critical points of f are the points $(1, 0)$ and $(-1, 0)$. Since f has continuous second partial derivatives we can use second derivative test to determine wether a critical point is a local maximum or minimum. We calculate

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y)$$

We have

$$f_{xx}(x, y) = 6x + 2y^2$$
$$f_{yy}(x, y) = 2x^2$$
$$f_{xy}(x, y) = 4xy$$

and therefore

$$D(x, y) = (6x + 2y^2)2x^2 - (4xy)^2 = 12x^3 + 4x^2y^2 - 16x^2y^2$$

We have

$$D(1, 0) = 12,\ f_{xx}(1, 0) = 6 > 0 \Longrightarrow (1, 0) \text{ is a local minimum.}$$
$$D(-1, 0) = -12 < 0 \Longrightarrow (-1, 0) \text{ is a saddle point.}$$

Version B. Find and classify (i.e., local maximum, local minimum, saddle point, or inconclusive) all critical points of the function:

$$f(x, y) = y^3 - \frac{3}{2}y^2 + \frac{3}{2}x^2 - 3xy + 5.$$

Solution. Solve $f_x = 3x - 3y = 0$ and $f_y = 3y^2 - 3y - 3x = 0$. The first equation gives $x = y$. The second gives $3y^2 - 6y = 0$ which implies that $y = 0$ or $y = 2$. Hence we have two critical points, $(0, 0)$ and $(2, 2)$.

At $(0, 0)$, compute $f_{xx} = 3$, $f_{xy} = -3$ and $f_{yy} = -3$ giving $D = f_{xx}f_{yy} - f_{xy}^2 = -9 - 9 = -18 < 0$. It follows that $(0, 0)$ is a saddle point.

At $(2, 2)$, compute $f_{xx} = 3$, $f_{xy} = -3$ and $f_{yy} = 6(3) - 3 = 15$. Then $D = f_{xx}f_{yy} - f_{xy}^2 = 45 - 9 = 36 > 0$ and since $f_{xx} > 0$ it follows that $(0, 0)$ is a local minimum.

7. Version A. Use the method of Lagrange multipliers to find the absolute maximum and absolute minimum values of the function $f(x, y) = xe^y$ on the curve given by $x^2 + y^2 = 2$.

Solution. Let $g(x, y) = x^2 + y^2$. Then $\nabla f = \langle e^y, xe^y\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. Setting $\nabla f = \lambda\nabla g$, we have the system of equations:
$e^y = \lambda 2x$

$xe^y = \lambda 2y$
$x^2 + y^2 = 2$
Notice that $x \neq 0$ and $\lambda \neq 0$, since otherwise by the first equation, we would have $e^y = 0$, which is impossible. Thus, we can divide the second equation by x to obtain $e^y = \frac{\lambda 2y}{x}$. Setting this equal to the first equation, we obtain $\frac{\lambda 2y}{x} = \lambda 2x$. Dividing by 2λ and multiplying by x, we obtain $y = x^2$. Plugging this into the third equation, we obtain $x^2 + x^4 = 2$ or $x^4 + x^2 - 2 = 0$. We can now solve this for x by factoring the left hand side into $(x^2 + 2)(x^2 - 1) = 0$. Thus either $x^2 = -2$ or $x^2 = 1$. The first equation has no solutions and the second equation has two solutions, namely $x = -1, 1$. The corresponding $y-$values are both 1 since $y = x^2$. Thus we have two points to compare, $(-1, 1)$ and $(1, 1)$. Plugging them into f, we obtain $f(-1, 1) = -e$ and $f(1, 1) = e$. Thus, the absolute maximum of f on the curve is e and the absolute minimum is $-e$.

Version B. Use the method of Lagrange multipliers to find the absolute maximum and absolute minimum values of the function $f(x, y) = e^{xy}$ on the curve given by $x^2+y^2 = 2$.

Solution. Let $g(x, y) = x^2 + y^2$. Then $\nabla f = \langle ye^{xy}, xe^{xy} \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. Setting $\nabla f = \lambda \nabla g$, we have the system of equations:
$ye^{xy} = \lambda 2x$
$xe^{xy} = \lambda 2y$
$x^2 + y^2 = 2$
Notice that $\lambda \neq 0$, since otherwise, by the first and second equations, we would have that $x = y = 0$, which contradicts the third equation. Moreover, $x \neq 0$, since otherwise, by the first equation, we would have that $y = 0$, once again contradicting the third equation. Similarly, $y \neq 0$. Thus, we can divide the first equation by y to obtain $e^{xy} = \frac{\lambda 2x}{y}$ and we can divide the second equation by x to obtain $e^{xy} = \frac{\lambda 2y}{x}$. Setting these equal, we obtain $\frac{\lambda 2y}{x} = \frac{\lambda 2x}{y}$. Cross multiplying and dividing by 2λ, we have $y^2 = x^2$. Plugging this into the third equation, we have $x^2 + x^2 = 2$. Solving for x, we obtain $x = -1, 1$. Since $y^2 = x^2$, the corresponding $y-$values for $x = -1$ are $y = -1, 1$ and the corresponding $y-$values for $x = 1$ are also $y = -1, 1$. Thus we have four points to compare, $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. Plugging them into f, we obtain $f(-1, -1) = e$, $f(-1, 1) = e^{-1}$, $f(1, -1) = e^{-1}$, and $f(1, 1) = e$. Thus, the absolute maximum of f on the curve is e and the absolute minimum is e^{-1}.

Solution.

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