Handout 5. Probability: Chapter 4.1-4.4

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Example:** Experiment 1: Toss a die,
  - Experiment 2: Toss two coins,
  - Experiment 3: Testing products as they come off an assembly line until we observe the defective product.
- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - One and only one simple event can occur when the experiment is performed.
  - A simple event is denoted by e with a subscript.
  - Each simple event will be assigned a probability, measuring "how often" it occurs.
- The set of all simple events (outcomes) of an experiment is called the **sample space**, S. An **event** is a collection of one or more simple events.

**Example 1 - The die toss:**

Sample space \( S = \{e_1, e_2, e_3, e_4, e_5, e_6\} \)

Define:
- Event A = "get 6" – simple event \( A = \{e_6\} \)
- Event B: an odd number, \( B = \{e_1, e_3, e_5\} \)
- Event C: a number > 2, \( C = \{e_3, e_4, e_5, e_6\} \)

B, C are not simple events.
- Two events are mutually exclusive (disjoint or incompatible) if, when one event occurs, the other cannot, and vice versa.
  (no elementary outcomes in common)

B and C are not disjoint events
A and B are disjoint events

**Experiment 1:** Toss a die
Sample space \( S = \{e_1, e_2, e_3, e_4, e_5, e_6\} \)

**Experiment 2:** Toss two coins.
Sample space \( S = \{HH, HT, TH, TT\} \).
- Event A: "get 2H" – simple event, \( A = \{HH\} \),
- Event B: "get at least one H" – not a simple event, \( B = \{HT, TH, HH\} \)

**Experiment 3:** Testing products as they come off an assembly line until we observe the defective product.
- S = \{D, GD, GGD, …\}, where D - defective product,
  - G - non-defective - has an infinite number of elements, but we can arrange them in a sequence (infinite countable).

Sample spaces from Exp1, Exp2, Exp3 – discrete sample spaces

**Experiment 4:** record the lifetime of a light bulb,
- Let \( t \) – time, then \( S = \{t: t \geq 0\} \) - has an infinite number of elements, we can not arrange them in a sequence – continuous sample space.

- The **intersection** of two events, \( A \) and \( B \), is the event that both \( A \) and \( B \) occur when the experiment is performed. We write \( AB \).
• The complement of an event $A$ consists of all outcomes of the experiment that do not result in event $A$. We write $\bar{A}$.

\[ \bar{A} \subseteq S \]

**Algebra of events**

• The union of two events, $A$ and $B$, is the event that either $A$ or $B$ or both occur when the experiment is performed. We write $A \cup B$.

\[ A \cup B \subseteq S \]

**Exercise 1.**

A fair coin is tossed three times, and events $A$ and $B$ are defined as follows:

A- at least 2 heads are observed
B- the number of heads observed is odd.

a) List all elementary events in the sample space of the experiment

b) Describe events $A, B$.

b) Find: $A \cup B, AB$ and complements of $A$ and $AB$.

c) Are the events $A$ and $B$ mutually exclusive?

**Interpretation of probability**

• One intuitive way of computing the probability of an event is by using the relative frequency (no knowledge, no assumptions)

– Suppose I can repeat an experiment an infinite number of times, always with the same conditions, and the result of the experiment is either the event $A$ or not $A$. ($A$ is a simple event)

– The probability of the event $A$ is:

\[ p(A) = \lim_{N \to \infty} \frac{n_A}{N} \]

Limit of the relative frequency as the sample size goes to infinity
Toss a coin 100 times, Head is 1, Tail 0. Event A-get a H 54 heads and 46 tails. Relative frequency = 0.54


- The probability of an event A is a measure of our belief that the event A will occur. One rule to compute probability is to use:

\[ P(A) = \frac{n_A}{N} \]

- Toss a die, 6 possible outcomes, numbers 1 to 6, so the probability that the upper face is 3 is 1/6.
- Toss a coin: two possible outcomes Head or Tail, so the probability of H is ½.

3. Subjective probabilities. P(A) is found by simply guessing or estimating its value based on knowledge of the relevant circumstances.

The Probability of an Event

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) = 1.
- The sum of the probabilities for all simple events in S equals 1.
  \[ P(S) = 1 \]

- The probability of an event A is found by adding the probabilities of all the simple events contained in A.

Exercise 2. Toss a fair coin twice. What is the probability of observing at least one head?

Calculating Probabilities for Unions and Complements

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
  - For any two events, A and B, the probability of their union, \( P(A \cup B) \), is
  \[ P(A \cup B) = P(A) + P(B) - P(AB) \]
  - We know that for any event A: \( P(A \bar{A}) = 0 \) and since either A or \( \bar{A} \) must occur, \( P(A \cup \bar{A}) = 1 \) so that \( P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1 \)
- **Law of Complement**
  \[ P(\bar{A}) = 1 - P(A) \]
A Special Case:

When two events A and B are mutually exclusive, 

\[ P(AB) = 0 \quad \text{and} \quad P(A \cup B) = P(A) + P(B). \]

Exercise 3

A sample space S consists of 4 simple events with these probabilities:

\[ P(e_1) = 0.15, \; P(e_2) = 0.2, \; P(e_3) = 0.4 \]

1) Find the probabilities for all simple events

2) Let \( A = \{ e_1, e_3 \} \) and \( B = \{ e_2, e_3 \} \)

Calculate

a) \( P(A) \), \( P(B) \),

b) \( P(\text{A does not occur}) \),

c) \( P(\text{at least one occurs}) \),

d) \( P(\text{both A and B occur}) \),

e) \( P(\text{Neither A nor B occurs}) \),

f) \( P(\text{A occurs and B does not occur}) \)

Exercise 4

If A, B are events with \( P(A) = 0.3 \), 

\( P(B) = 0.8 \) and \( P(AB) = 0.2 \),

calculate \( P(A \cup B) \), \( P(\bar{A} \cap B) \), \( P(A \text{ or } B \text{ but not both}) \)