Notes on regular unipotent and nilpotent elements

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To the memory of Bert Kostant

Here we summarize the basic facts about regular unipotent elements of a connected semisimple algebraic group G and regular nilpotent elements of its Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. Here we work with group schemes (or the groups of their rational points) over an algebraically closed field of arbitrary characteristic $p \geq 0$. Though most of these facts are by now well documented and can be stated (even if not always proved) uniformly, we also call attention to what is apparently unknown (especially in "bad" prime characteristics).

Note that groups G in the same isogeny class sometimes have different Lie algebras when $p > 0$: details are summarized in papers based on Hogeweij's thesis [4] and in [5] for type A_ℓ when $p|(\ell + 1)$. In practice this has relatively litte effect in our situation because the differences are concentrated in a Cartan subalgebra (the Lie algebra of a maximal torus in G); but there is sometimes a difference in the resulting centralizers. We assume for convenience that G is *simply connected*; then it is well known that $\mathfrak{g}$ is the Chevalley Lie algebra over K coming from a Chevalley $\mathbb{Z}$-form of a corresponding semisimple Lie algebra over $\mathbb{C}$. In the work of both Springer and Steinberg, analogous $\mathbb{Z}$-forms lead to the Lie algebras of all G.

1 Preliminaries

Assume from now on that the root system Φ of G is irreducible (i.e., G is simple as an algebraic group) and fix a system of simple roots. First we recall that a *good prime* p is one which fails to divide any coefficient of the highest root of G when this root is written as a $\mathbb{Z}^+$-linear combination of simple roots. One sees from the classification that $p = 2$ is *bad* (not good) for all types except A_ℓ, while $p = 3$ is bad just for exceptional types and $p = 5$ just for type E_8. (See [18, I, 4.3] or [6, 3.9]. Since the Lie algebra $\mathfrak{sl}_n$ of rank $n-1$ has a nontrivial center just when $p|n$, we also say that p is *very good* if p is good but does not divide $\ell + 1$ in type A_ℓ. Finally, the field K is said to be of *good characteristic* if its characteristic is 0 or else a good prime $p > 0$. Such distinctions are crucial in most treatments of structure theory, e.g., [3, 12, 15].

Two irreducible closed subsets in (respectively) G and $\mathfrak{g}$ are key players: the variety $\mathcal{U}$ of all unipotent elements of G and the variety $\mathcal{N}$ of all nilpotent elements of $\mathfrak{g}$. These are both of dimension equal to $\dim G - \ell$, where G has rank ℓ. (This is stated briefly in [1, 1.15], but the proof for bad p depends on the existence of regular nilpotent elements in $\mathfrak{g}$.) In good characteristic there is always an isomorphism between these two varieties, equivariant relative to the natural actions of G. In characteristic 0 this is given uniformly by the inverse maps exp and log, while for good $p > 0$ there is a more complicated method originally due to Springer which also involves the normality of the variety $\mathcal{N}$. (See for example [6, Chap. 6] and [7, §2].)

Each of the two varieties is in a natural way a disjoint union of finitely many conjugacy classes (in the case of $\mathcal{U}$) or $\operatorname{Ad} G$-orbits (in the case of $\mathcal{N}$). While this follows readily from the classification of weighted Dynkin diagrams for nilpotent orbits in characteristic 0, the proof in characteristic $p > 0$ is less direct (for $\mathcal{U}$ and $\mathcal{N}$ separately).

Following work of Richardson for most p, a more comprehensive but subtle argument by Lusztig showed the finiteness of the number of unipotent classes in $\mathcal{U}$. But case-by-case study has been needed for $\mathcal{N}$ when p is bad (see [1, 7.15; Chap. 5] or [6, 3.8, 3.11]. An unavoidable complication is that the classes and orbits are not always in bijection when p is bad. However, there are always just finitely many of them, and their dimensions are always even. Moreover, there always turns out to be just one dense class or orbit of *regular* elements.

2 Regular elements of G and of $\mathfrak{g}$

Here we recall briefly some basic facts about regularity. Following the work of Kostant [9, 10] in characteristic 0 (mainly in the setting of complex semisimple Lie algebras and their adjoint groups), major progress was made by Steinberg [19, 20] and Springer [16, 17] in the setting of semisimple algebraic groups and their Lie algebras. For modern expositions, often incorporating the later Bala–Carter approach to classification theory in good characteristics, see for example [1, Chap. 5], [6, Chap. 4], [18, III].

It is easy to see that the centralizer of any element of G has dimension at least ℓ (see for example [6, 1.6]). By definition, an element $x \in G$ is *regular* if its centralizer $C_G(x)$ has this least possible dimension ℓ. Similarly, an element $z \in \mathfrak{g}$ is called *regular* if its "centralizer" (isotropy group under $\operatorname{Ad} G$) $C_G(z)$ in G has the least possible dimension ℓ. This is the definition given by Springer [16, 5.7] and used by Keny [8]. But in some sources $C_G(z)$

is replaced by the Lie algebra centralizer $\mathfrak{c}_{\mathfrak{g}}(z)$. In fact the two definitions are the same unless $\mathfrak{g}$ has a nontrivial center, possible only if p is not very good; cf. [16, 5.9].

We remark that the definitions extend readily to *reductive* groups such as $G = \mathrm{GL}_n(K)$ and its Lie algebra $\mathrm{M}_n(K)$. In this concrete case, the notion of regularity for a square matrix is familiar in matrix theory under the old-fashioned name *non-derogatory*, which translates into the condition that the characteristic polynomial of an $n \times n$ matrix be equal to the minimal polynomial. (For a nilpotent matrix, this just means that the Jordan normal form of the matrix consists of a single block.)

The intrinsic Jordan–Chevalley decomposition $x = su$ in G (product of commuting semisimple and unipotent parts) often allows one to reduce questions inductively to the regular unipotent case. This uses the fact that $C_G(s)$ is reductive since s is semisimple (and even connected if G is simply connected, by arguments of Springer and Steinberg), while in turn the unipotent part u of x lies in the semisimple derived group of $C_G(s)$.

Remark. One elegant way to characterize regular elements in G is that $x \in G$ is regular if and only if x lies in just finitely many Borel subgroups. For example, if $x = s$ is regular and semisimple, it lies in exactly $|W|$ Borel subgroups, where W is the Weyl group. At the other extreme, if $x = u$ is regular and unipotent, then it lies in a unique Borel subgroup.

It is reasonable to expect a parallel characterization of regular elements in $\mathfrak{g}$ as those which lie in only finitely many Borel subalgebras. For this one should define a *Borel subalgebra* of $\mathfrak{g}$ to be the Lie algebra of a Borel subgroup of G (rather than say as a maximal solvable subalgebra). In this way one could exploit the fact that for all $p \geq 0$ the Borel subalgebras are in natural bijection with the Borel subgroups: cf. [5, §14]. One then wants to characterize the regular nilpotent elements as those lying in a unique Borel subalgebra (cf. [16, 5.3]).

3 Regular unipotent elements

At first the most difficult task is to show that regular unipotent elements always exist.

Theorem A. *Let G as above be a connected semisimple algebraic group over an algebraically closed field K of arbitrary characteristic $p \geq 0$. Then:*

(a) *Regular unipotent elements u exist in G.*

(b) *Such elements form a single conjugacy class in G, dense in $\mathcal{U}$.*

(c) *For each fixed choice of basis in Φ, the product of corresponding unipotent elements for arbitrarily chosen nontrivial simple root group elements is a regular unipotent element. Having nontrivial components for all simple roots in fact characterizes those regular unipotent elements which lie in the unipotent radical of the Borel subgroup corresponding to the choice of simple roots.*

Kostant's treatment of complex semisimple Lie algebras in [9] achieved similar results relative to what he called *principal* nilpotent elements of $\mathfrak{g}$. Here the adjoint Lie group plays the role of G. (In his 1963 paper [10] the more general concept of regular element comes into play in the guise of a set denoted $\mathfrak{r}$ but without the name "regular". Here he shows, for example, in Prop. 14 that the centralizer of any element in $\mathfrak{r}$ is abelian.)

A uniform algebraic group treatment in aritrary characteristic was soon provided by Steinberg [19], while at about the same time Springer was obtaining case-by-case results under the assumption that p is a good prime [16]. A major difference in their approaches involved the proof of existence in (a). For this Steinberg observed that the double coset BwB for a fixed Borel subgroup B and a *Coxeter element* w of the Weyl group (product of simple reflections) must consist of regular elements including a unipotent one. Springer's method was more concrete, involving the Chevalley commutation relations (modulo p) in the unipotent radical of B. This approach was later used effectively for bad p to obtain more detail, in case-by-case computations for G and $\mathfrak{g}$ respectively by two students of Steinberg, Lou [14] and Keny [8] as recalled below.

It was understood by both Springer and Steinberg that the type of unipotent element described in (c) should be regular, but even in special cases it isn't usually straightforward to describe its centralizer explicitly (and thus compute the dimension).

Remark. In characteristic $p > 0$, any unipotent element of G has order equal to some power of p. Using the Bala–Carter method of classification, Testerman [21, 0.4] shows how to compute in exceptional types (for good p) the order of a *distinguished* unipotent and thus all orders. Recalling that the Coxeter number h is 1 plus the height of the highest positive root, the order of a *regular* unipotent element u is shown to be the least power $p^a \geq h$. For example, in type E_8 the regular unipotents have order p when $p > 30 = h$, while the order is actually p^2 for the good primes $p = 7, 11, 13, 17, 19, 23, 29$.

Precise details for specific primes in all exceptional cases and for all p were worked out by Lawther [11], who gave tables of the Jordan blocks for a class representative in low-dimensional representations. For example, his Table 9 contains the data for type E_8 (later slightly corrected to reflect problems with Mizuno's older classification); here the class of regular unipotents is also labelled E_8 and the adjoint module of dimension 248 gives the smallest faithful representation of G. (For respective bad primes $p = 2, 3, 5$ one then sees from the Jordan blocks that the orders are given respectively by $2^5, 3^4, 5^3$.)

4 Regular nilpotent elements

Theorem B. *With G as above a connected semisimple algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$, in any characteristic $p \geq 0$ we have:*

(a) *Regular nilpotent elements e exist in $\mathfrak{g}$.*

(b) *Such elements e form a single orbit under $\mathrm{Ad}\, G$, dense in $\mathcal{N}$.*

(c) *For each fixed choice of simple roots, the sum of corresponding arbitrarily chosen nonzero root vectors is a regular nilpotent element. Indeed, regular nilpotent elements in the nilradical of the corresponding Borel subalgebra are characterized by having nonzero components for each simple root.*

For good primes this is due to Springer in [16, §5]. Having proven the facts summarized in Theorem A above for good p, he uses parallel arguments for $\mathfrak{g}$. To deal with bad primes, Keny [8] proceeds case-by-case using his approach, by a combination of general arguments for classical types and computer methods for exceptional types.

5 Centralizers and isotropy groups

When $x = s$ is *semisimple* and regular, its centralizer $C_G(s)$ is just a maximal torus and thus is connected. But when $x = u$ is *unipotent* and regular, the situation is more complicated: $Z(G)$ (which is finite and consists of semisimple elements) obviously lies in $C_G(u)$. For each fixed regular unipotent element $u \in G$ and regular nilpotent element $e \in \mathfrak{g}$, most (but not quite all) details concerning the structure of the centralizer $C_G(u)$ and the isotropy group $C_G(e)$ have been worked out by now.

Theorem C. *Take G and $\mathfrak{g}$ as above.*

(a) *Let $u \in G$ be a regular unipotent element, lying in the unipotent radical U of the unique Borel subgroup B containing it. Then $C_G(u)$ is the direct product of $Z(G)$ and $C_U(u)$. Moreover, $C_U(u)$ is connected precisely when p is good. If p is bad, the component group $C_U(u)/C_U(u)^\circ$ is cyclic, generated by the coset of u.*

(b) *Let $e \in \mathfrak{g}$ be any regular nilpotent element. Then $C_G(e)$ is the direct product of $Z(G)$ and $C_U(e)$. Moreover, $C_U(e)$ is always connected.*

(c) *The group $C_G(u)$ is abelian, of dimension ℓ.*

(a) This is due to Springer [16, 4.11, 4.12], for good and bad p respectively. When p is bad, $C_G(u)$ is always disconnected even when $Z(G)$ is trivial; here u fails to live in the identity component $C_G(u)^\circ$, but its coset generates the cyclic component group $C_U(u)/C_U(u)^\circ$.

(b) If p is good, this follows from Springer's results for G by using his equivariant isomorphism between $\mathcal{U}$ and $\mathcal{N}$ (cf. [18, III, 3.7]). For bad p, Keny's case-by-case proof of the existence of regular nilpotent elements yields indirectly this structure of the isotropy group. The tables in Liebeck–Seitz [13] show the equality of dimensions of $C_G(u)$ and $C_G(e)$ as well as the fact that $C_U(e)$ is always connected (confirming Keny's work).

(c) In the case of G, the fact that $C_U(u)^\circ$ is abelian follows from Springer's general result [17] that the centralizer of any element of G contains an abelian subgroup of dimension ℓ (the argument is also given in [6, 1.14]). But he left the question whether $C_U(u)$ itself is abelian open for bad p.

Subsequent case-by-case work by Lou in [14] for bad p shows that the component group is always cyclic, generated by the coset of u; thus $C_U(u)$ is always abelian, forcing $C_G(u)$ to be abelian as well. Note however (as Liebeck and Seitz realized while computing their tables in [13]) that Lou erred in concluding that the component group in type F_4 when $p = 2$ has order 2. This component group is actually cyclic of order 4, as Lou had shown for the exceptional types $\mathrm{E}_7, \mathrm{E}_8$ (when $p = 2$). In all other cases, the component group is cyclic of order p.

What about $C_G(e)$ when $e \in \mathfrak{g}$ is a regular nilpotent element? It is natural to expect that it too will always be abelian, as it is for good p thanks to the existence of Springer maps between $\mathcal{U}$ and $\mathcal{N}$. From Keny's work (or that of Liebeck–Seitz) we know that $C_U(e)$ is connected.

In view of (c), one natural way to prove the abelian property would be to show that the two connected groups $C_U(u)^\circ$ and $C_U(e)^\circ = C_U(e)$ of dimension ℓ are always conjugate, and even equal for appropriate choices of u and e. This is true in characteristic 0 or in good characteristic $p > 0$, where the unipotent and nilpotent varieties are isomorphic in a G-equivariant way; moreover, the Lie algebra $\operatorname{Lie} C_U(u)$ contains a regular nilpotent element of $\mathfrak{g}$. However, for bad p the action of $\operatorname{Ad} u$ on $\operatorname{Lie} C_U(e)$ might be nontrivial, even though the conjugation action of u on $C_U(u)^\circ$ is trivial.

Here and at some earlier points of the development, we still lack "natural" uniform proofs (especially for results obtained only by case-by-case checking). So there is some work to be done.

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