Longest element of a finite Coxeter group

September 10, 2015

Here we draw together some well-known properties of the (unique) longest element $w_\circ$ in a finite Coxeter group W, with reference to theorems and exercises in several older books [1, 2, 3]. For the most part these properties can be proved without case-by-case use of the well-known classification of the groups W, but we also point out a few examples based on the classification. The longest element plays an important role in many contexts, including for example the determination of conjugacy classes of involutions in Coxeter groups by Richardson and Springer.

Historically, Weyl groups gave rise to a number of the properties and their applications, independently of the somewhat more general geometric treatment due especially to Coxeter. We mention Lie-theoretic connections at the end of the note. On the other hand, the history of the subject is quite hard to disentangle.

Fix a finite irreducible Coxeter group W relative to a set S of n involutive generators. Here n is called the *rank*; it is often denoted by symbols such as r, l, ℓ. To avoid trivialities here, we assume $n > 1$. There is an associated "root system" (in Deodhar's sense) which we denote by Φ, along with a choice of "simple reflections" $s_1, \dots, s_n$ in W in natural bijection with S. Here the roots are $\pm$ unit vectors orthogonal to the reflecting hyperplanes for all reflections in W, in a suitable real euclidean space V on which W acts faithfully. (The length convention here is contrary to the refined notion of root system used in Lie theory.) To this data there corresponds a linearly independent set of n simple roots Δ as well as sets of positive and negative roots Φ^+, Φ^-. Set $N := |\Phi^+|$.

As Coxeter showed, the abstract groups W arising in this way are (up to isomorphism) precisely the finite irreducible reflection groups acting on real euclidean spaces. They have a well-known classification as *crystallographic* types $A_n, B_n = C_n, D_n, E_6, E_7, E_8, F_4, G_2$ (for which any product of two distinct reflections has order $2, 3, 4$ or 6) along with *non-crystallographic* types $H_3, H_4, I_2(m)$. For example, the groups W of type A_n are just the symmetric groups S_{n+1} (acting faithfully by permutation of coordinates on the hyperplane of euclidean $(n+1)$-space where coordinates sum to 0) and the groups of type $I_2(m)$ are the dihedral groups of order $2m$. Overlap is avoided by restricting the rank n: for example, the label D_n is used just when $n \geq 4$, while the groups $I_2(m)$ are non-crystallographic just when $m \neq 1, 2, 3, 4, 6$. The crystallographic reflection groups W coincide with the Weyl groups attached to simple complex Lie algebras as in [2]; cf. [1, VI, 2.5] and [3, 2.9].

Here we mainly follow the notation in [3], which differs somewhat from that in [1]. These two sources differ much more substantially in their logical arrangement, since the treatise [1] begins with the general theory of Coxeter groups whereas the textbook [3] emphasizes first the concrete study of finite and affine real reflection groups.

1 The longest element of W

Early in the general development, one can prove (using the finiteness of W) the existence of a unique element $w_\circ \in W$ sending Φ^+ to the set of negative roots. Then $w_\circ$ has order 2 and has length $\ell(w_\circ) = N$. Here the *length* $\ell(w)$ of any $w \in W$ is defined as usual to be the least number of simple reflections s_i occurring in any expression for w (then the expression is called *reduced*). Moreover, any reduced expression for $w_\circ$ must involve all of the simple reflections $s_1, \dots, s_n$. These facts are developed somewhat differently in [1, V, §4, Exer. 2b), VI, 1.6, Cor. 3 of Prop. 17] and [3, 1.8]; both use essentially the fact that W is simply transitive on positive systems of roots.

2 Coxeter elements of W

To get more insight into $w_\circ$ one needs to invoke the notion of *Coxeter element* in W. This is one of the deeper aspects of Coxeter's geometric study of finite reflection groups but has only limited impact on the general study of Coxeter systems in [1]. Initially the idea is fairly simple: enumerate the elements of S in some order as $s_1, \dots, s_n$ and define $w := s_1 \cdots s_n$. At this stage one appeals to the elementary fact that the Coxeter graph associated to the pair (W, S) has no circuits: an easy first step in the direction of a full classification of finite Coxeter groups (or Weyl groups). From this fact it follows readily that all orderings of S define conjugate elements in W, so any one of these can be taken to be a Coxeter element. Now define the *Coxeter number* h of W to be the order of any Coxeter element. (For example, when $W = S_{n+1}$, its rank is n and its Coxeter elements are the $(n+1)$-cycles, so $h = n + 1$.)

It is convenient to make a special choice of w, as follows. The fact that the Coxeter graph of W has no cycles implies that the vertices (simple reflections) can be divided into two nonempty disjoint subsets $S' := \{s_1, \dots, s_r\}$ and $S'' := \{s_{r+1}, \dots, s_n\}$ so that in each subset the reflections all commute.

The respective products y and z over S', S'' are therefore involutions, whose product is taken to be w. (If $W = S_{n+1}$ we can take S', S'' to be the respective sets of transpositions $(1,2),(3,4),\dots,$ and $(2,3),(4,5),\dots$)

Even after the classification is in hand, it remains a challenging problem to compute the Coxeter numbers h explicitly. For this, Coxeter's more subtle analysis comes into play. This is developed in similar ways (but with different notation) in [1, Chap. V] and in [3, Chap. 3]. The main idea is to locate a special plane P in the n-dimensional space where w acts, so that P meets both the fundamental chamber C and the opposite one $w_\circ C$: their walls lie in the hyperplanes corresponding to S. The action of w on this plane is just a rotation of order h. In more detail the two specially chosen lines which span P define naturally a dihedral subgroup of W of order $2h$, with y and z acting as reflections in the respective lines and their product w generating the rotation subgroup. Then $C \cap P$ consists of all linear combinations of y, z with strictly positive coefficients. Moreover, each reflecting hyperplane for W intersects P in one of the w-rotated lines. (It then follows that every Weyl chamber in V meets P nontrivially. This is not so easy to visualize; for this it would be instructive to study types A_3, B_3, H_3.)

In the references cited, the Coxeter number is treated (partly for reasons which are made explicit below) in the context of the action of W on polynomials arising from the given euclidean space. Here Chevalley showed that the algebra of W-invariants is itself a polynomial algebra in the same number of indeterminates (a generalization of the usual theorem on symmetric functions for the case $W = S_{n+1}$ acting on euclidean n-space). Moreover, the degrees d_i of such generators are uniquely determined by W and are called the *degrees* of W, the least being 2 and the largest being h (for example, $2, \dots n+1$ when $W = S_{n+1}$).

3 A basic relationship among some constants.

By making essential use of Coxeter's set-up just sketched, one can deduce an elegant relationship among three fundamental constants already defined:

Theorem. [1, V,6.2,Thm. 1], [3, 3.18] *With notation as above,* $2N = nh$.

It follows in particular that at least one of the two numbers n and h must be *even*. This is illustrated again by $W = S_{n+1}$, which has rank n. As observed above its Coxeter elements are the $(n+1)$-cycles, while $h =$

$n+1$. The number N of positive roots always agrees with the number of reflections in W (here corresponding to transpositions), which in this example is $n(n+1)/2$.

Once the complete list of finite irreducible Coxeter groups has been worked out, the basic constants already mentioned here can be found in various sources. For example, see the *planches* for the crystallographic root systems in [1], or the tables giving for all types the degrees [3, 3.7] amd the constants h, n, N [3, 3.18]. (For Weyl groups see also [2, §12].)

4 When is -1 in W?

Viewed concretely as a finite reflection group acting faithfully on a euclidean space V, W often (but not always) turns out to contain the orthogonal transformation -1; this is then the default choice for $w_\circ$. Precise conditions for -1 to lie in W are most conveniently based on the properties of canonical degrees and Coxeter elements in W outlined above. Apart from the fact that the Coxeter graph of W has no circuits, this is done independently of the classification in [1, V, 6.2, Cor. 2 of Prop. 3] or [3, 3.7]:

Proposition. $-1 \in W$ *if and only if all* d_i *are even; then* $w_\circ = -1$.

One knows that h is the largest of the degrees $d_1, \ldots, d_n$. (We will exhibit below an easy way to produce a reduced decompositon of $w_\circ$ whenever h is even.) When one works out the d_i and h for each W using the classification, it turns out more precisely that $-1 \notin W$ precisely in the crystallographic types A_n (for $n > 1$), D_n (for $n \geq 5$ odd), E_6 and in the non-crystallographic types $I_2(m)$ (for $m \geq 5$ odd). (Note that $A_3 = D_3$ in the classification.)

5 A reduced decomposition of $w_\circ$?

For any group such as W given by generators and relations, one might ask for a list of all possible reduced expressions of a particular element such as $w_\circ$ in terms of the given generators $s_1, \ldots, s_n$. This may of course lead to an unrealistically long list, but it is at least reasonable to look for a single specific expression. When the Coxeter number h is *even*, this can readily be supplied in a classification-free manner:

Proposition. *If* h *is even and a Coxeter element* w *is constructed as above, then* $w_\circ = w^{h/2}$.

This is proved in [1, V, 6.2, Prop. 2] (under the assumption, omitted in the first printing, that W is irreducible). It is also outlined as an exercise in [3, Exer. 3.19] (under the implicit assumption, added in the list of revisions, that w has the special form chosen in 3.17). The idea is to observe from the action of w on the Coxeter plane P that $w^{h/2}$ takes points of the fundamental chamber C in P to points of the opposite chamber $w_\circ C$ lying in P. In view of the fact that $\overline{C}$ is a fundamental domain for the action of W on V, while points of C have trivial stabilizers in W, this implies $w^{h/2} = w_\circ$.

In turn, it follows immediately from our expression of $w_\circ$ as a product of $(nh)/2$ simple reflections that $N = \ell(w_\circ) \leq (nh)/2$. But by the theorem in §3, we have $N = (nh)/2$; so the expression here for $w_\circ$ is in fact reduced.

What happens when h is odd? Using the classification of finite Coxeter groups and case-by-case determination of their Coxeter numbers, one sees that the only cases when h is odd are types A_n with n even (while $h = n+1$) and type $I_2(m)$ (with $m = h$ odd but $n = 2$). The latter is a dihedral group and easy to study directly. For the symmetric group S_{n+1} of type A_n with n even, one can see directly that a modification of the above procedure is enough to produce a reduced decomposition: here $w_\circ$ is the product of $w^{n/2}$ times the first factor y of w when w is written as a product of two involutions (or equally well as the product of the second factor z of w times $w^{n/2}$.

A general argument along this line (not using the classification) is outlined in [1, V, 6.2, Exer. 2b].) As above, the cases when h is odd or even are necessarily treated separately. Starting with h odd, Bourbaki suggests the following steps. Construct w as before as a product of two involutions: $w = yz$, with $y = s_1 \cdots s_r$ and $z = s_{r+1} \cdots s_n$ for a suitable numbering of simple reflections and for some $r < n$. Then observe that $w^{(h-1)/2}\ y = z\ w^{(h-1)/2}$. This is an easy exercise, grouping the alternate factors y and z in two ways and using the fact that y, z have order 2. Finally, use the action on P of the dihedral group generated by y, z to argue that either expression represents $w_\circ$ because it sends points of $C \cap P$ to points of $w_0 C \cap P$. Again as a consequence of the theorem in §3, it follows that at least one of the expressions is reduced; in turn, $r = n/2$ (so in fact both expressions are reduced).

We remark that combinatorists including Stanley and Stembridge have been able to work out case-by-case the reduced expressions for $w_\circ$ (or at least count the total number of these) for many types. Much of this work grew out of Stanley's classical 1984 paper referenced in [3].

6 Special case: Weyl groups

The history of the subject is complicated by the fact that the theories of Cartan–Weyl and Coxeter developed separately and only sporadically intersected (due to observations of Witt, Coxeter, and others). So the basic ideas about *Weyl groups* of simple Lie algebras over $\mathbb{C}$ (or compact simple Lie groups) are often derived in Lie theory textbooks such as [2] independently of the somewhat more general theory of finite Coxeter groups: in the latter theory, Weyl groups appear as the crystallographic Coxeter groups.

The longest element $w_\circ$ of a Weyl group W occurs frequently in Lie theory, especially in representation theory. For example, given a finite dimensional simple module $L(\lambda)$ for a simple Lie algebra having highest weight λ (a dominant integral linear function on a Cartan subalgebra), the dual module $L(\lambda)^*$ has highest weight $-w_\circ\lambda$ (which is always dominant and agrees with λ at least when $-1 \in W$). See for example discussions of $w_\circ$ in [2, Exer. 10.9, Exer. 12.6, Exer. 13.5, Exer. 21.6], as well as [1, VI, 1.6, Prop. 17, Cor. 3].

More recently, Lusztig's work on quantum groups and canonical bases has drawn attention to the need to work with a specified reduced expression for $w_\circ$ when choosing positive (but not simple) root vectors. On the other hand, Coxeter elements of W arise only indirectly in Lie theory but (as seen above) they lead elegantly to some of the more subtle features of $w_\circ$.

References

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV–VI, Hermann, Paris, 1968.

2. J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972; revisions posted at http://people.math.umass.edu/˜jeh/pub/lie.pdf

3. ——, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, 1990; revisions posted at http://people.math.umass.edu/˜jeh/pub/cox2.pdf