Fundamental Modules for Simple Algebraic Groups

(March 3, 2014; revised with appendix added June 15, 2016)

Here we gather information from a number of sources about Weyl modules $V(\lambda)$ and their simple quotients $L(\lambda)$ when the highest weight λ is one of the fundamental weights for a simple, simply connected algebraic group G over an algebraically closed field of arbitrary characteristic $p \geq 0$. We refer to the $V(\lambda)$ or $L(\lambda)$ in such cases as *fundamental modules*.

Of course, when the characteristic is 0, we have $V(\lambda) = L(\lambda)$ for all dominant weights λ, and in this case the dimensions are given by Weyl's formula. When $p > 0$, $\dim L(\lambda)$ is sometimes known explicitly when it is less than $\dim V(\lambda)$, as is some information about the composition factors of $V(\lambda)$. For the exceptional types (here including D_4), dimensions of such $L(\lambda)$ for most of the fundamental weights λ are given below in tables, based on the more detailed computations of Frank Lübeck.

Remark: While the determination of dimensions and formal characters for arbitrary $L(\lambda)$ is far from complete, a number of special cases have been worked out. For example, recent work by Peter Sin [10] for type E_6 focuses on restricted multiples $r\varpi_1$ (with $0 \leq r < p$). In another direction, the adjoint module (for G or its Lie algebra $\mathfrak{g}$) has been studied for all primes by Hiss and Hogeweij. For more detailed results, see [6, II.8.21].

1 Notation

For the most part we follow Jantzen [6] or for root systems [4, 5]. Here we always take G to be a simply connected algebraic group of rank ℓ over an algebraically closed field K of arbitrary characteristic $p \geq 0$. The root system of G is taken to be irreducible, thus of type $\mathrm{A}_\ell, \dots, \mathrm{G}_2$. with fixed simple roots $\alpha_1, \dots, \alpha_\ell$ and Weyl group W.

Fundamental dominant weights $\varpi_1, \dots, \varpi_\ell$ form a natural $\mathbb{Z}$-basis of the full weight lattice Λ, in which the root lattice Λ_r with basis $\alpha_1, \dots, \alpha_\ell$ has finite index. The set of dominant weights is denoted Λ^+.

2 Minuscule Weights

Beyond the trivial 1-dimensional module $V(0) = L(0)$, the easiest case to deal with involves a nonzero highest weight λ for which the only weights of $V(\lambda)$ are the W-conjugates of λ; this forces $V(\lambda) = L(\lambda)$. Such weights are called by Bourbaki *minuscule* and must be among the fundamental weights. Together with 0, such weights are precisely the *minimal* dominant weights in the usual partial ordering of Λ. These minimal weights in fact provide coset representatives for all cosets of Λ/Λ_r.

References include [2, VIII, §7, no. 3] (cf. [1, VI, §1, Exer. 23], [1, VI, §2, Exer. 5]), [4, Exer. 13.13]. (See also the book by Green [3], which develops combinatorial aspects.) We summarize in a table the list of minuscule weights for each simple type (there being none in types $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8$).

type	weights
A_ℓ	$\varpi_1, \ldots, \varpi_\ell$
B_ℓ	ϖ_ℓ
C_ℓ	ϖ_1
D_ℓ	$\varpi_1, \varpi_{\ell-1}, \varpi_\ell$
E_6	ϖ_1, ϖ_6
E_7	ϖ_7

Table 1: Minuscule weights

3 Classical Groups

As the list indicates, all fundamental Weyl modules in type A_ℓ are simple.

In type C_ℓ (symplectic groups), the literature exhibits most of the submodule structure of fundamental Weyl modules. While $V(\varpi_\ell)$ is simple, other cases involve more complicated structure. A brief survey, with references, is given in [5, 4.5]. The main results are due to Premet–Suprunenko, McNinch, Gow. and Foulle.

The story is less complete in types $\mathrm{B}_\ell, \mathrm{D}_\ell$ (special orthogonal groups and spin coverings), but only for $p = 2$ can a fundamental Weyl module fail to be simple: for this and further references see [6, II.8.21]. For type D_4 explicit results are given in the table below.

Remark. When $p = 2$, Chevalley found a "special isogeny" between the simply connected algebraic groups of types B_ℓ and C_ℓ which induces an isomorphism (as abstract groups) of the groups of rational points over K. Steinberg refined this to show that the rational representations of these algebraic groups are essentially the same. This shows up for $p = 2$ in the low-rank tables of Lübeck [8]. There are also special isogenies (from the group to itself) for G_2 when $p = 3$ and for F_4 when $p = 2$, as one can observe in the tables below. (See [5, 5.3–5.4].)

This accounts for the exceptions in an otherwise general theorem of Premet [9]: if p exceeds all ratios of squared lengths of roots (which may be $1, 2, 3$) and $\lambda \in \Lambda^+$ is p-restricted, then $L(\lambda)$ has the same subweights as $V(\lambda)$ but with possibly smaller multiplicity. Note that F_4 in characteristic 2 actually satisfies this conclusion, but just barely.

4 Tables for Exceptional Groups

The tables here provide explicit but not quite complete information about dimensions of fundamental Weyl modules and their simple quotients as the characteristic varies. Here $\dim V(\lambda)$ is known from Weyl's formula (see [2, VIII, Table 2] or tables at the end of [11]), while $\dim L(\lambda)$ has been worked out in some special cases over many years. At first such computations were done by hand, for instance by Veldkamp and Springer in type F_4. But later on, work by Burgoyne and (in a more efficient version) by Gilkey-Seitz took advantage of computer techniques.

The computations by Lübeck [7, 8] extend the range of such results considerably and include the data presented here. In most cases the highest weights studied by him are arbitrary, but upper bounds are imposed on the dimensions (which grow very rapidly). In our tables, which include very recent work by Lübeck, only a few cases in type E_8 remain to be completed for all exceptional primes. All cases where the dimension of some $L(\lambda)$ is smaller than $\dim V(\lambda)$ are listed explicitly in the tables; otherwise $L(\lambda) = V(\lambda)$.

Wherever possible, we have compared different sources to reinforce confidence in the results assembled here. Note that we regard type D_4 as "exceptional". Here three of the four fundamental weights are minuscule, those permuted nontrivially by an automorphism of the Dynkin diagram. See Lübeck's online tables [8] for a listing of dominant subweights (with multiplicities). These weights may (for some primes) yield highest weights of proper composition factors of $V(\lambda)$. The theorem of Premet mentioned above actually shows that for any p all dominant subweights do occur with positive multiplicity in $L(\lambda)$ with few exceptions; but the weight multiplicities in $L(\lambda)$ may be smaller than those in $V(\lambda)$ for some $p > 0$. This is seen in Lübeck's detailed tables.

Concerning type E_8, the classical dimensions of fundamental Weyl modules vary radically. Only three fall below the bound imposed in Lübeck's tables. The smallest dimension is 248 when $\lambda = \varpi_8$, which is also the dimension of $L(\lambda)$ for all p. This is just the *adjoint representation* on the Lie algebra $\mathfrak{g}$ of G. Note that $V(\lambda)$ is not minuscule, since it involves the dominant subweight 0 (whose multiplicity 8 remains the same for $L(\lambda)$).

In our tables, the numbering of fundamental weights (or vertices of the Dynkin diagram) conforms to that in Bourbaki [1], or [4]. But there are variations in many sources.

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	7	6 ($p=2$)
ϖ_2	14	7 ($p=3$)

Table 2: Data for G_2

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	8	
ϖ_2	28	26 ($p=2$)
ϖ_3	8	
ϖ_4	8	

Table 3: Data for D_4

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	52	26 ($p=2$)
ϖ_2	1274	246 ($p=2$)
		1222 ($p=3$)
ϖ_3	273	246 ($p=2$)
		196 ($p=3$)
ϖ_4	26	25 ($p=3$)

Table 4: Data for F_4

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	27	
ϖ_2	78	77 ($p = 3$)
ϖ_3	351	324 ($p = 2$)
ϖ_4	2925	1702 ($p = 2$)
		2771 ($p = 3$)
ϖ_5	351	324 ($p = 2$)
]ϖ_6	27	

Table 5: Data for E_6

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	133	132 ($p = 2$)
ϖ_2	912	856 ($p = 3$)
ϖ_3	8 645	7 106 ($p = 2$)
		6 512 ($p = 3$)
ϖ_4	365 750	110 790 ($p = 2$)
		295 679 ($p = 3$)
		364 211 ($p = 13$)
ϖ_5	27 664	21 184 ($p = 2$)
		25 896 ($p = 3$)
ϖ_6	1 539	1 274 ($p = 2$)
		1 538 ($p = 7$)
ϖ_7	56	

Table 6: Data for E_7

References

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV–VI, Hermann, 1968.
2. ——–, *Groupes et algèbres de Lie*, Chap. VII–VIII, Hermann, 1975.
3. R.M. Green, *Combinatorics of minuscule representations*, Cambridge Tracts in Mathematics, 199. Cambridge Univ. Press, 2013.
4. J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.
5. ——–, *Modular Representations of Finite Groups of Lie Type*, Cambridge Univ. Press, 2006.
6. J.C. Jantzen, *Representations of Algebraic Groups*, 2nd ed., Amer. Math. Soc., 2003.

weight	$\dim V(\lambda)$	$\dim L(\lambda)$
ϖ_1	3875	3626 ($p = 2$)
ϖ_2	147 250	except $p = 2, 3, 7$
ϖ_3	6 696 000	except $p = 2, 3, 19$
ϖ_4	6 899 079 264	except $p = 2, 3, 5, 13, 19$
ϖ_5	146 325 270	except $p = 2, 3, 5$
ϖ_6	2 450 240	except $p = 2, 3, 5, 7$
ϖ_7	30 380	26 504 ($p = 2$)
		30 132 ($p = 3, 5$)
ϖ_8	248	

Table 7: Data for E_8

7. F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

8. ——–, Online tables posted at:
www.math.nwth-aachen.de/˜Frank.Luebeck/chev/WMSmall/index.html

9. A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Mat. Sb. (N.S.) **133** (1987), no. 2, 167–183; translation in Math. USSR-Sb. **61** (1988), no. 1, 167-183.

10. P. Sin, *Some Weyl modules of the algebraic groups of type* E_6, pp. 279–300 in: Groups of exceptional type, Coxeter groups and related geometries, Springer Proc. Math. Stat., 82, Springer, New Delhi, 2014.

11. J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lect. Notes in Math. 40, Springer, 1967.

5 Appendix

Here are several complements to the previous exposition, along with a supplementary list of references.

1) Concerning the classical origins of weights (including the fundamental ones), probably the most useful early source is a paper by Élie Cartan [Ca13] in which he worked out case-by-case descriptions of the weights of finite dimensional representations. Note that he used notation ϖ (a variant of π closer to handwriting) for a typical weight (or *poids*). This symbol was taken over by Bourbaki [1, 2] (though it is frequently confused with the greek letter ω), who also standardized the symbol ρ for the sum of all fundamental weights.

In the study of finite dimensional representations of semisimple Lie algebras (say over $\mathbb{C}$), an important advance was made by Freudenthal in a three part paper, cf. [Fr54]. He worked out the first reasonably efficient recursive algorithm for computing weight multiplicities. An exposition is given in various textbooks including [4]. He relied on Cartan's foundational work but corrected in Tafel E of his part II one tiny error in Cartan's tables. Modulo this correction, the older results computed by hand have held up well over the years, though modern computer methods allow one to treat explicitly many more high dimensional representations.

2) In the process of studying fundamental dominant weights, one usually encounters the highest root $\widetilde{\alpha}$, which is the highest weight of the *adjoint representation* of each Chevalley Lie algebra. Case-by-case examination of the *planches* at the end of [1] reveals that this is usually a fundamental weight, except in types A_ℓ and C_ℓ. Using Bourbaki's numbering, we find that $\widetilde{\alpha} = \varpi_2$ in types B_ℓ, D_ℓ, E_6, G_2, but instead $= \varpi_1$ in types E_6, F_4, and $= \varpi_8$ in type E_8.

In type A_ℓ with $\ell \geq 2$, we get instead $\widetilde{\alpha} = \varpi_1 + \varpi_\ell$, while in type C_ℓ with $\ell \geq 2$ we get $2\varpi_1$ (as we do for A_1, which should perhaps be labelled as type C_1).

In general, the dominant weight 0 occurs in the adjoint module, with multiplicity ℓ, and in all cases except E_8 there is some p for which the trivial module occurs as a composition factor. The proof of this goes back to older work by Dieudonné [Di57], Steinberg [St61] but is recovered in later computations. There is a detailed account for example in Hogeweij's 1978 thesis [Ho78], cf. [Ho82]. In his own thesis work, Hiss [Hi84] studied carefully the structure of the adjoint module for the corresponding adjoint group.

Earlier thesis work in [Hu67], usually under the restriction $p > 3$, showed in detail how the Lie algebras of G and its quotients by various subgroups of the center behave in characteristic p. The least obvious case involves type A_ℓ with $p|(\ell + 1)$; here the extent to which p divides the index $\ell + 1$ of the root lattice in the weight lattice determines which of three possible structures the Lie algebra has.

3) The authors of a recent preprint [GGN16] take a different viewpoint, suggested by the work of B. Gross on "global irreducibility" for finite groups: Starting with an irreducible finite dimensional Lie algebra representation of a given dominant highest weight λ in characteristic 0, they use Chevalley's process of reduction modulo each prime p to obtain a Weyl module $V(\lambda)$ giving a module for G which may or may not be simple. The problem is to sort out for which primes p one gets a simple module, thus deciding for which dominant weights λ all $V(\lambda)$ are simple. They work out a complete solution using diverse techniques.

Soon afterward Jantzen [Ja16] found a more uniform proof using elementary roots-and-weights methods, in the process of answering a question raised at the end of [GGN16]: *Given a dominant weight λ and a dominant subweight μ which is maximal among all dominant subweights, when does the simple G-module $L(\mu)$ of highest weight μ occur as a composition factor of $V(\lambda)$?* As noted earlier, when G has type E_8 and λ is the highest root $\widetilde{\alpha}$, the weight $\mu = 0$ never yields such a composition factor. Otherwise Jantzen shows that $L(\mu)$ does occur as a composition factor for some p whenever μ exists (i.e., λ is neither minuscule nor 0).

Added References

[Ca13] É. Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane*, Bull. Soc Math. France **41** (1913), 53–96; *Oeuvres Complète*, Tome I, Partie I, pp. 355–398, CNRS, Paris, 1984.

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