Revisiting Brauer's formula for tensor product decompositions

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A classical problem in Lie theory is to decompose a tensor product of finite dimensional simple modules into a direct sum of simple modules (possible in principle from Weyl's complete reducibility theorem), in an explicit way:

$$(*) \quad L(\lambda) \otimes L(\mu) \cong \bigoplus_{\nu} c^{\nu}_{\lambda\mu}\, L(\nu).$$

The notation used here will be explained below. A familiar (though oversimplified) example is given for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by the Clebsch–Gordan formula; as in [2, Exer. 22.7]. In 1937 Richard Brauer published a short note giving a general formula of this sort. It still serves as the starting point for some computer methods, even though it usually involves a large number of cancellations.

Here our purpose is to revisit Brauer's formula and related matters from the perspective of the BGG (Bernstein-Gelfand-Gelfand) category $\mathcal{O}$ attached to a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field (or other splitting field) of characteristic 0. These ideas from the early 1970s provide new insights into the finite dimensional Cartan–Weyl theory by working also with certain infinite dimensional modules (see [3] for a recent account). Our approach was suggested by J.C. Jantzen. He, along with Allen Knutson and Shrawan Kumar, also provided valuable comments on an early version of this note.

It is difficult to say what might constitute the "simplest" or most transparent proof of Brauer's formula, since by now the tensor product decomposition has been studied using many tools ranging from Lie algebra theory to algebraic geometry and combinatorics. Only a few references are included below. Apparently Brauer's formula depends essentially on the Weyl character formula, but first we discuss some more elementary steps leading to qualitative estimates about the possible summands in the decomposition $(*)$.

1 Notation and background

Over the past century the notational conventions in Lie theory have varied widely. Here we rely mainly on the notation used in [3], which draws on many sources. The structure of the Lie algebra $\mathfrak{g}$ depends mainly on its root system Φ and Weyl group W relative to a fixed Cartan subalgebra $\mathfrak{h}$. Given a simple system Δ in Φ, positive and negative roots $\alpha \in \Phi$ are defined, as well as a length function $\ell(w)$ on W.

There are $|W|$ Weyl chambers in a real vector space spanned by Φ. Denote by Λ the lattice of integral weights in this vector space. Relative to Δ, the *dominant* integral weights Λ^+ lie in the dominant Weyl chamber, whose closure is a fundamental domain for the natural action of W.

In category $\mathcal{O}$ the simple modules with integral weights (the only ones we consider) are parametrized by highest weights $\lambda \in \Lambda$. Typically these are infinite dimensional. For each $\lambda \in \Lambda$ there is a universal highest weight module (Verma module) $M(\lambda) \in \mathcal{O}$ and a canonical surjection $M(\lambda) \to L(\lambda)$. The finite dimensional simple $\mathfrak{g}$-modules are those $L(\lambda)$ for which $\lambda \in \Lambda^+$. In this case W permutes the weights of $L(\lambda)$, preserving the multiplicity $m_\lambda(\mu) = \dim L(\lambda)_\mu$ in each orbit. Denote by $\Pi(L(\lambda))$ the set of all weights μ for which $m_\lambda(\mu) > 0$.

Each module $M \in \mathcal{O}$ with integral weights has a *formal character* given by $\operatorname{ch} M = \sum_{\mu\in\Lambda} \dim M_\mu \, e(\mu)$, a formal sum recording weight multiplicities.

All modules in $\mathcal{O}$ have finite length. Highest weights of composition factors $L(\mu)$ of $M(\lambda)$ are linked to λ by the *dot-action* of W, given by $\mu = w\cdot\lambda := w(\lambda+\rho)-\rho$. Here ρ is the sum of fundamental dominant weights (relative to Δ), equal to the half-sum of positive roots. After shifting the origin for the W-action to $-\rho$, each $\mu \in \Lambda$ is linked by the dot-action to an element of $\Lambda^+ - \rho$ for a unique $w \in W$.

2 Possible summands and multiplicities

Using just basic facts about the BGG category $\mathcal{O}$ for $\mathfrak{g}$, we can get some qualitative estimates on the decomposition $(*)$. (This was worked out first by Kostant [7], as noted by Kumar [9, (3.2)].)

(A) Fix $\lambda \in \Lambda$, $\mu \in \Lambda^+$. In a direct sum decomposition of $L(\lambda) \otimes L(\mu)$ into simple modules $L(\nu)$ (with $\nu \in \Lambda^+$), the only possible ν which can occur have the form $\lambda + \pi$ for some $\pi \in \Pi(L(\mu))$. Moreover, $L(\nu)$ occurs in such a direct sum at most $m_\mu(\pi)$ times.

Using the exactness of tensoring with a fixed finite dimensional module in $\mathcal{O}$, we obtain from the canonical map $M(\lambda) \to L(\lambda)$ a surjection $\varphi : M(\lambda)\otimes L(\mu) \to L(\lambda)\otimes L(\mu)$. On the left side there exists a *standard* filtration having as subquotients the Verma modules $M(\lambda+\pi)$ with multiplicity $m_\mu(\pi)$ as π runs over $\Pi(L(\mu))$. This elementary result of BGG is developed in two ways in [3, 3.6].

The composition factors of $M(\lambda)\otimes L(\mu)$ are therefore those of the Verma modules $M(\lambda + \pi)$ taken with the indicated multiplicities. Passing to the quotient $L(\lambda) \otimes L(\mu)$ retains only some finite dimensional composition factors. We claim these can occur only as top composition factors $L(\lambda + \pi)$. For this it is most transparent to apply the results of Verma and BGG [3, 5.1]: unless the highest weight of a Verma module is *dominant*, no composition factor can be finite dimensional. This is clear from "strong linkage", because the dominant Weyl chamber lies on the positive side of all reflecting hyperplanes. Moreover, in case the highest weight $\lambda + \pi$ belongs to Λ^+. no strongly linked lower weight can be dominant. (But some finite dimensional composition factors might remain in the kernel of the map φ, due to unknown extensions among the Verma modules in a standard filtration.)

Combined with the above description of multiplicities in a standard filtration of $M(\lambda) \otimes L(\mu)$, both assertions of (A) follow—but not yet an explicit formula.

3 Brauer's formula

While (A) limits the possible simple summands of $L(\lambda) \otimes L(\mu)$ and bounds their multiplicities, an explicit formula requires a more sophisticated approach. This was first realized by Brauer [1] in 1937, though his short note has some rough spots. (Weyl gives an account in [13, VII.10]; see his note 22. A quick proof of the formula was obtained by Jantzen [4, p. 447] as a consequence of his Lemma 8.) A method like Brauer's was later developed by A.U. Klimyk, adapted to the computational needs of physicists.

Brauer's basic idea is fairly simple (and is easily pictured in rank 1 or 2 cases): start with the full weight diagram of $L(\mu)$, which is W-symmetric around 0, then translate this diagram in the dominant direction by adding λ to all weights π. In case the resulting weights all lie in Λ^+, Brauer's formula shows that these are precisely the highest weights (taken with multiplicity $m_\mu(\pi)$) of the various simple modules $L(\nu)$ occurring in the decomposition of the given tensor product. (As Kostant notes in [7], this is just the case when μ is *totally subordinate* to λ in Dynkin's terminology.) At the other extreme, one might consider (perhaps perversely) the case when $\lambda = 0$.

In general some of the weights $\nu = \lambda + \pi$ will fail to be dominant. To deal with these we add a bit of notation. For any $\nu \in \Lambda$, there is a unique $w \in W$ taking ν to a weight $\nu' \in \Lambda^+$ by the dot-action: $\nu' = w \cdot \nu$. If $\nu' \notin \Lambda^+$ (meaning it has a coordinate equal to -1), set $\chi(\nu) := 0$. Otherwise define $\chi(\nu) := (-1)^{\ell(w)} \operatorname{ch}(\nu')$, where $\operatorname{ch}(\nu')$ is the formal character of the simple module $L(\nu')$. With this notation, we can state Brauer's formula:

$$\text{(B)} \quad \mathrm{ch}\,\big(L(\lambda)\otimes L(\mu)\big) = \sum_{\pi\in\Pi(L(\mu))} m_\mu(\pi)\,\chi(\lambda+\pi).$$

For the proof, we again work in category $\mathcal{O}$. There the module $L(\lambda)$ has a *BGG resolution* [3, 6.1], by the modules $\bigoplus_{\ell(w)=i} M(w\cdot\lambda)$, which in effect realizes the Kostant form of Weyl's character formula (as discovered by BGG). Since tensoring in $\mathcal{O}$ with a finite dimensional module is an exact functor, this resolution yields a corresponding resolution of the module $L(\lambda)\otimes L(\mu)$ by direct sums of modules $M(w\cdot\lambda)\otimes L(\mu)$. This allows us to express $\mathrm{ch}\,\big(L(\lambda)\otimes L(\mu)\big)$ as an alternating sum of formal characters of all the Verma modules occurring in standard filtrations of the latter tensor products in $\mathcal{O}$:

$$(**) \quad \mathrm{ch}\,\big(L(\lambda)\otimes L(\mu)\big) = \sum_{w\in W}(-1)^{\ell(w)}\sum_{\pi\in\Pi(L(\mu))} m_\mu(\pi)\;\mathrm{ch}\,M(w\cdot\lambda+\pi).$$

Some bookkeeping is needed in order to organize the double sum on the right side in accordance with Brauer's formula. For this, observe first that by definition $w\cdot\lambda+\pi = w\cdot(\lambda+\pi')$, where $\pi' = w^{-1}\pi$ is again a weight of $L(\mu)$ with the same multiplicity. Using this for each fixed w, we can replace $M(w\cdot\lambda+\pi)$ by $M\big(w\cdot(\lambda+\pi)\big)$ on the right side of $(**)$:

$$(***) \quad \mathrm{ch}\,\big(L(\lambda)\otimes L(\mu)\big) = \sum_{w\in W}(-1)^{\ell(w)}\sum_{\pi\in\Pi(L(\mu))} m_\mu(\pi)\;\mathrm{ch}\,M\big(w\cdot(\lambda+\pi)\big).$$

Now fix an arbitrary weight π of $L(\mu)$ and set $\nu := \lambda+\pi$. We dispose first of the case when ν is "dot-irregular", thus fixed by some reflection $s_\alpha\in W$. Write the sum over W in $(***)$ as a sum over pairs $\{w, ws_\alpha\}$ of formal characters of Verma modules. Here the formal characters coincide but the signs cancel, leaving only $0=\chi(\nu)$ as desired.

When ν is "dot-regular", there are two cases:

(1) $\nu\in\Lambda^+$. The formal characters (with alternating sign) of the Verma modules $M(w\cdot\nu)$ involved in a BGG resolution of $L(\nu)$ are already displayed on the right side of $(***)$, with multiplicity $m_\mu(\pi)$. This agrees with the definition of $\chi(\nu)$.

(2) $\nu\notin\Lambda^+$. In this case, there is a unique $y\in W$ for which $y\cdot\nu\in\Lambda^+$. Using the fact that wy runs over W as w does, we see that the right side of $(***)$ involves $(-1)^{\ell(y)}$ times the formal character coming from a BGG resolution of $L(y\cdot\nu)$ in the situation of (1). This is again consistent with the way $\chi(\nu)$ was defined.

4 Some other approaches to tensor products

By now a variety of approaches have been developed to deal with the classical tensor product decomposition $(*)$ and its offshoots. We mention some influential directions, along with a few influential papers.

In type A_n the finitely many nonzero coefficients $c^{\nu}_{\lambda\mu}$ in $(*)$ are called *Littlewood–Richardson coefficients*. Special or general linear Lie algebras and the associated groups over $\mathbb{C}$ have especially attracted the attention of combinatorists. Here W is a symmetric group,while weights can be thought of as partitions. After the initial development of the L–R rule for computing the coefficients, newer approaches have been found to this and related problems: see for instance the work of Knutson and Tao [6].

For arbitrary semisimple Lie algebras $\mathfrak{g}$, the work of Littelmann on "paths" has drawn considerable attention from combinatorists, since it arrives at numerical results in tensor product computations without doing elaborate cancellations. See especially his paper [10].

Older work by Partharasathy, Ranga Rao, and Varadarajan led to the *PRV Conjecture* which predicts that certain highest weights. related to extremal weights of one of the modules being tensored, must occur with positive multiplicity in $(*)$. This resisted purely algebraic methods for a long time, but eventually the conjecture was proved in a precise form using algebraic geometry: see the independent work of Kumar [8] (cf. [9]) and Mathieu [11]. The geometry arises naturally from study of flag varieties and the like attached to an algebraic group having Lie algebra $\mathfrak{g}$. (Earlier papers by Kempf also explored tensor products, in terms of the realization of a simple module as the global sections of a line bundle on the flag variety.)

References

1. R. Brauer, *Sur la multiplication des caractéristiques des groupes continus et semi-simples*, C.R. Acad. Sci. Paris **204** (1937), 1784–1786.
2. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.
3. J. E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*, Amer. Math. Soc., 2008.
4. J.C. Jantzen, *Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie–Algebren*, J. Algebra **49** (1977), 441–469.
5. J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lect. Notes in Math., Springer, 1979..

6. A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055-1090.
7. B. Kostant, *A formula for the multiplicity of a weight*, Trans. Amer. Math. Soc. **93** (1959), 53–73.
8. S. Kumar, *Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture*, Invent. Math. **93** (1988), 117-130.
9. S. Kumar, *Tensor product decomposition*, Proc. ICM, v. III, 1226-1261, Hindustan Book Agency, New Delhi, 2010.
10. P. Littelmann, *Paths and root operators in representation theory,*Ann. of Math. (2) **142** (1995), 499-525.
11. O. Mathieu, *Construction d'un groupe de Kac-Moody et applications*, Compositio Math. **69** (1989), 37-60.
12. R. Steinberg, *A general Clebsch–Gordan theorem*, Bull. Amer. Math. Soc. **67** (1961), 406–407.
13. H. Weyl, *The Classical Groups*, 2nd ed., Princeton Univ. Press, 1946.