Notes on D_4

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Consider the simple Lie algebra $\mathfrak{g}$ of type D_4 over an algebraically closed field K of characteristic $p > h = 6$ (the Coxeter number). In particular, p is a *good* prime. We have $\dim \mathfrak{g} = 28$, with $N = 12$ positive roots, while the Weyl group W has order 192.

Here we assemble some details about three aspects of $\mathfrak{g}$ which are known or conjectured to be closely related (see [7] and references there): nilpotent orbits; cells in a corresponding affine Weyl group; representations (not necessary restricted) of $\mathfrak{g}$ over K attached to nilpotent orbits.

First we summarize concisely in a table known data in the case of D_4, followed by remarks and references to sources. Many ideas are expected to carry over to other semisimple Lie algebras if the characteristic is good (or sometimes greater than the Coxeter number): then the nilpotent variety of $\mathfrak{g}$ and unipotent variety of its adjoint group can be identified in such a way that the essential properties of nilpotent orbits and unipotent classes agree. Here are some abbreviations used in the table:

$\mathcal{O}_e$	Orbit of given $e \in \mathcal{N}$ (= nilpotent variety of $\mathfrak{g}$)
d	Half-dimension of $\mathcal{O}_e$
$A(e)$	Component group $A(e) := C_G(e)/C_G(e)^\circ$ if nontrivial
$\mathcal{B}_e$	Fiber over e in Springer's desingularization of $\mathcal{N}$, identified with the set of Borel subalgebras containing e
a	$\dim \mathcal{B}_e$ (= Lusztig's a-invariant of cell Ω_e associated to $\mathcal{O}_e$)
IC	Number of irreducible components of $\mathcal{B}_e$
R	Is $\mathcal{O}_e$ a Richardson orbit?
S	Is $\mathcal{O}_e$ special?
LT	Is $\mathcal{O}_e$ of standard Levi type (e is regular in some Levi subalgebra)?
M	Number of simple modules in a regular block of the reduced enveloping algebra attached to e
LC	Number of left cells in the two-sided cell Ω_e

orbit $\mathcal{O}_e$	d	$A(e)$	a	IC	R	S	LT	M	LC
$[7,1]$	12		0	1	Y	Y	Y	1	1
$[5,3]$	11		1	4	Y	Y	N	5	5
$[5,1^3]$	10		2	3	Y	Y	Y	8	8
$[4^2]^{\mathrm{I}}$	10		2	3	Y	Y	Y	8	8
$[4^2]^{\mathrm{II}}$	10		2	3	Y	Y	Y	8	8
$[3^2,1^2]$	9	$\mathbb{Z}_2$	3	14	Y	Y	Y	32	22
$[3,2^2,1]$	8		4	2	N	N	Y	24	24
$[3,1^5]$	6		6	3	Y	Y	Y	48	48
$[2^4]^{\mathrm{I}}$	6		6	3	Y	Y	Y	48	48
$[2^4]^{\mathrm{II}}$	6		6	3	Y	Y	Y	48	48
$[2^2,1^4]$	5		7	4	N	Y	Y	96	96
$[1^8]$	0		12	1	Y	Y	Y	192	192

Table 1: Data for type D_4

Sources and remarks

(1) The nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$ comprises 12 nilpotent orbits relative to the adjoint group G. Labels for orbits involve partitions of 8, since the natural module for $\mathfrak{g}$ has dimension 8. The partial ordering of orbits by inclusion of one orbit in the closure of another is the most obvious ordering compatible with dimensions of orbits. Data about the orbits is assembled in the table, along with data on representations attached to an orbit and on the left cells of the associated 2-sided cell of the affine Weyl group of D_4 (relative to Lusztig's bijection).

(2) The nilpotent orbits of D_4 have been well-studied: see for example [3, 5, 13]. In [3] see pp. 426–427, 449. In [5] see pp. 84, 97, 103, 118; but note that the orbit with partition $[3^2,1^2]$ is mistakenly omitted on p. 84. In two cases there is a triple of orbits with the same dimension; these also share other data, since an outer automorphism of $\mathfrak{g}$ of order 3 permutes them naturally.

(3) The Weyl group W of type D_4 has also been well-studied, together with its characters (in the general setting of type D_n). Here W is a semidirect product of S_4 with an elementary abelian group of order 8, the latter being normal. Thus $|W| = 192 = 2^6 \cdot 3$. The set $\widehat{W}$ of its

characters has 13 elements, of degrees $1, 1, 2, 3, 3, 3, 3, 3, 3, 4, 4, 6, 8$; the character of degree 6 is not Springer and that of degree 2 is nonspecial in Lusztig's sense. The characters are realized by Springer theory: in case the component group $A(e)$ is trivial, the top cohomology of the Springer fiber $\mathcal{B}_e$ affords an irreducible character of W having degree equal to the number IC of irreducible components of $\mathcal{B}_e$. (Values of IC for type D_4 are found in [22, p. 239].) When $\mathcal{O}_e$ has type $[3^2, 1^2]$ and $A(e) = \mathbb{Z}_2$, the dimension of the top cohomology is 14 and affords the character of degree 8 along with the character of degree 6 tensored with the nontrivial character of $\mathbb{Z}_2$.

(4) Lusztig conjectured and later proved (by using deep geometric methods) that there is a bijection between nilpotent orbits $\mathcal{O}_e$ of $\mathfrak{g}$ (or rather unipotent classes of G) and 2-sided cells Ω_e in the (dual) affine Weyl group: see [15] and the references there. In his bijection, the a-invariant of Ω_e agrees with the dimension of the Springer fiber $\mathcal{B}_e$ for a typical e in the corresponding orbit. The values range from $a = 0$ for the regular orbit to $a = N$ for the zero orbit. Here $d = N - a$ is half the dimension of the orbit, as seen in the table. Lusztig also conjectured that his bijection respects the natural partial orderings on cells and on orbits. This was shown by Shi in rank < 5 and in type A, then (much less directly) by Bezrukavnikov in general.

(5) In a 1983 paper, Lusztig [14, 3.6] conjectured that the number of left cells in the two-sided cell Ω_e is given in good characteristic by

$$\sum_{i \geq 0} (-1)^i \dim H^i(\mathcal{B}_e, \overline{\mathbb{Q}}_\ell)^{A(e)}.$$

This has not yet been proved in general. He formulated the conjecture for unipotent elements and arbitrary p, but it carries over to nilpotents in good characteristic. In that case all odd degree cohomology is known to vanish, so the sum gives the dimension of the fixed point space of $A(e)$ on the total cohomology of the Springer fiber $\mathcal{B}_e$. (The dimension of this total cohomology is computable in most cases using Lusztig's induction theorem [20] for Springer representations.)

(6) Column LC in the table specifies the number of left cells in each correlated 2-sided cell. This is worked out in the special case D_4 indepen-

dently by Chen [4] and Shi [21], based on similar combinatorial techniques. Although direct computations are intricate and hard to double-check, the results here agree with Lusztig's conjecture just quoted.

(7) The zero orbit corresponds to restricted representations of $\mathfrak{g}$, coming from representations of a simply connected group of the same type: see [11]. For $p \geq h$, Lusztig's 1980 conjecture should provide recursively the dimensions and formal characters of simple modules in this case. This is not yet proved in full generality, but in any case the number of simple modules in each regular block for $\mathfrak{g}$ is given uniformly by $|W|$.

(8) For background on the non-restricted representations of $\mathfrak{g}$, see [6]; many details are worked out by Jantzen [8, 9, 10, 12]. Simple modules attached by Kac–Weisfeiler to nilpotent orbits are the crucial ones to understand. As they conjectured and Premet proved (under mild restrictions), all $\mathfrak{g}$-modules for a given orbit of dimension $2d$ have dimensions divisible by p^d.

(9) A nilpotent orbit $\mathcal{O}_e$ has "standard Levi form" if e is regular in some Levi subalgebra of $\mathfrak{g}$, say determined by a subset I of simple reflections in W. In this case the number M of simple modules in a regular block is always $|W|/|W_I|$, where W_I is the subgroup of W generated by I (Friedlander–Parshall). More detailed information predicted by Lusztig [17] is verified by Jantzen in special cases (some unpublished).

(10) For the regular orbit (here $d = 12$), a regular block has only one simple module and its dimension is p^d. The subregular case ($d = 11$) is worked out for D_4 and most other cases by Jantzen [9]. In unpublished work on type D_n he also gives details about one orbit with $d = 10$ (including the number M and explicit dimension formulas). Much less is known about dimensions for nonzero orbits of D_4 with $d \leq 9$.

(11) For D_4 the number M can be computed by using the algebraic results just quoted. In general the work of Bezrukavnikov, Mirković, and Rumynin [1, 2] has shown for $p > h$ that the number M is given by the dimension of the total cohomology of the associated Springer fiber $\mathcal{B}_e$.

(12) Lusztig's proposed formalism [15, §10] for the asymptotic Hecke algebra associated with a 2-sided cell Ω_e of the affine Weyl group is expected to be modeled by the set of simple modules in a regular block of the

corresponding reduced enveloping algebra for a simple Lie algebra (of dual type): see [19, 7] and forthcoming joint work of Bezrukavnikov and Mirković. Here each $A(e)$-orbit in the set of M simple modules in a regular block should be assigned uniquely to a left cell. (The numbers here make sense in view of Lusztig's approach [14] to counting left cells, combined with the result of [1] just quoted and the equivariance of their category equivalences relative to $A(e)$. This insures that $A(e)$ acts on the total cohomology of $\mathcal{B}_e$ by a permutation representation, forcing the number of orbits in the set of simple modules to agree with the dimension of the fixed point space.) For D_4, the nilpotent orbit of type $[3^2, 1^2]$ has component group $\mathbb{Z}_2$ acting with 22 orbits in all: 12 singletons and 10 pairs, in a natural bijection with the 22 left cells.

(13) In general it is reasonable to ask when a higher power of p than p^d can divide one or more dimensions of simple modules in a regular block attached to a nilpotent orbit of dimension $2d$. The relatively few examples known so far from Jantzen's work (in rank ≤ 3 or involving "small" blocks) behave consistently: In each instance there is a *special piece* of $\mathcal{N}$, involving a special orbit $\mathcal{O}_e$ together with one or more smaller nonspecial orbits in its closure; then $A(e) \neq 1$ according to Lusztig [16, Thm. 0.4]. Two or more simple modules attached to $\mathcal{O}_e$ form an $A(e)$-orbit, with a common dimension of the form $p^d m$ (p not dividing m). These "degenerate" to a single module of the same dimension attached to a nonspecial orbit; such a pattern might be repeated in passing to a smaller nonspecial orbit, leading again to a higher than expected p-power in some dimension there. So far it is precisely for nonspecial orbits that examples are known where an unexpected p-power occurs; is this a general fact?

(14) In the case D_4 there is a special piece involving the special orbit $[3^2, 1^2]$ with $d = 9$ and $A(e) = \mathbb{Z}_2$, together with the nonspecial orbit $[3, 2^2, 1]$ with $d = 8$. It would be especially interesting to compute the p-powers dividing dimensions here. From the cell data one expects to have 10 pairs of simple modules attached to $[3^2, 1^2]$, each pair sharing a dimension $p^9 m$ and retaining this dimension under degeneration to a single module attached to the orbit $[3, 2^2, 1]$. (Recent calculations by Jantzen, based in part on Lusztig's conjecture in [17], confirm this expectation while exhibiting closed formulas for many dimensions of simple mod-

ules.)

(15) In the general setting, the limited evidence available so far suggests a strong correlation between nonspecialness of nilpotent orbits and occurrence of higher than expected p-powers in dimensions. This raises natural questions about two cases in which nilpotent orbits are always special: type A_ℓ and the zero nilpotent orbit. For example, are there any simple restricted modules in regular blocks which have dimensions divisible by p?

It is a consequence of Lusztig's conjecture on characters in the restricted case that (for p large enough) the dimensions of simple modules are $\mathbb{Z}$-linear combinations of Weyl dimensions with coefficients independent of p. For p-regular weights the Weyl dimensions in question are not divisible by p and are all congruent up to sign modulo p, using Weyl's formula and linkage under the affine Weyl group. From this and the format of Lusztig's conjecture it would follow that most p do not divide the dimensions of simple modules in a regular block. (The question can also be asked about quantum groups for the relevant weights and primes. Here the analogue of Lusztig's conjecture is known to be true.)

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