In problems that require reasoning, algebraic calculation, or the use of your graphing calculator, it is not sufficient just to write the answers. You must explain how you arrived at your answers and show all your algebraic calculations.

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1. [18 points] [VASIN] (a) [9 pts] During a week, the proportion of time $X$ that a machine is down for maintenance or repair has a Beta distribution with $\alpha = 1$ and $\beta = 2$. That is

$$f(x) = \begin{cases} 
2(1-x), & 0 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}$$

The cost of this down time due to lost production and maintenance is given by $C = 10 + 3X + 6X^2$. Find the expectation and variance of $C$.

Solution:

$$E[X] = \int_0^1 2(1-x) \, dx = 1 - \frac{2}{3} = \frac{1}{3}$$

$$E[X^2] = \int_0^1 2(1-x)^2 \, dx = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$E[X^3] = \int_0^1 2(1-x)^3 \, dx = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$E[X^4] = \int_0^1 2(1-x)^4 \, dx = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$$

Four expectations, 1.5 pts each.

**E[C] = 10 + 3E[X] + 6E[X^2] = 10 + 1 + 1 = 12.**

1 pt for $E[C]$


$$V[C] = E[C^2] - (E[C])^2 = 147.5 - 144 = 3.5$$

2 pts for $V[C]$.

(b) Let $X$ be an exponential random variable with parameter $\lambda = 2$, and let $Y$ be the random variable defined by $Y = 8e^X$. Compute the (cumulative) distribution function of $Y$, namely $F(y) = P(Y \leq y)$.

Solution: Since $X \sim Exp(2)$, the probability density function of $X$ is given by

$$f_X(x) = \begin{cases} 
2e^{-2x}, & 0 \leq x < \infty, \\
0, & \text{elsewhere}
\end{cases}$$
3 pts for the density function.

We compute the distribution for $Y$:

$$F_Y(y) = P(Y \leq y) = P\left(X \leq \ln\left(\frac{y}{8}\right)\right) = F_X\left(\ln\left(\frac{y}{8}\right)\right)$$

$$= \int_0^{\ln\left(\frac{y}{8}\right)} 2e^{-2x} dx = 1 - \frac{64}{y^2}, \quad y \in (0, \infty).$$

Using directly the formula for the distribution (CDF) of the exponential, is also acceptable.

6 pts for the final result. Deduct one point if the student does not know $e^{\ln x} = x$. 
2. [18 points] [PANAGIOTA] (a) [9 points] Assume $X$ is a normal random variable with mean 20 and variance 16, and $Y$ is a Gamma random variable with parameters 5 and 2. In addition $X$ and $Y$ are independent. Construct a box with length $L = |X|$, width $W = 2|X|$, and height $H = Y$. Let $V$ be the volume of the box. Calculate the expected value $E[V]$?

Solution: $V = L \cdot W \cdot H$.

1 point.

$$E[V] = E[2|X| \cdot |X| \cdot Y] = 2E[X^2 \cdot Y] = 2E[X^2] \cdot E[Y].$$


3 pts for $E[X^2]$  

$$E[Y] = 5 \cdot 2 = 10.$$  

Therefore, $E[V] = 8320$. 2 pts for the final result.

(b) [9 points] If $a$ is uniformly distributed over $[-11, 21]$, what is the probability that the roots of the equation

$$x^2 + ax + a + 15 = 0$$

are both real?

Solution: Since $a \sim \text{Uniform}(-11, 21)$, the probability density function of $a$ is given by:

$$f_a(\alpha) = \begin{cases} \frac{1}{32}, & x \in [-11, 21], \\ 0, & \text{elsewhere.} \end{cases}$$

2 pts for density function.

For the real roots of the given equation, $a$ should satisfy the following

$$a^2 - 4a - 60 \geq 0$$

2 points for the inequality

which implies that

$$\{a \leq -6\} \ \text{OR} \ \{a \geq 10\}$$
3 points for finding the correct interval. 

Therefore,

\[ P(\{a \leq -6\} \text{ or } \{a \geq 10\}) = P(\{a \leq -6\}) + P(\{a \geq 10\}) \]

\[ = \int_{-6}^{-1} \frac{1}{32} \, d\alpha + \int_{1}^{10} \frac{1}{32} \, d\alpha = \frac{1}{2} \]

2 points for the final result.
3. [10 points] [YAO] A certain brand of cereal has an average shelf life of 7 months. Assume that based on the current technology used in the manufacturing and packaging processes the standard deviation from its’ average shelf life is $\sigma = 1$ month.

(a) [5 points] Use Tchebysheff’s Theorem to determine whether it is likely (probability of occurrence $> 66\%$) that the contents of a cereal box will stay fresh anywhere between 5 and 9 months

(b) [5 points] What is the desired standard deviation if the percentage in (a) needs to improve to at least 99%?

Solution: (a) Let the random variable $X$ be the shelf life of the cereal. 5 months is 2 standard deviations to the left of the mean; 9 months is 2 standard deviation to the right of the mean. Using the Tchebyshev’s Theorem, see Theorem 4.13. We have $P(-2\sigma \leq X - \mu \leq 2\sigma) \geq 3/4 = 75\%$.

3 points for formulating the problem to the correct form. 2 pts for applying Chebyshev’s inequality.

(b) Use Tchebysheff’s Theorem, see Theorem 4.13. $1 - \frac{1}{k^2} = .99$, thus $k = 10$. Then we need $k\sigma \leq 2$, thus $\sigma = .2$ months or less.

2 points for formulating the problem to the correct form. 1 pts for finding $k = 10$. 2 pts for applying Chebyshev’s inequality.
4. [18 points] [JIANYU] Suppose that $X \sim \text{Uniform}(0,1)$. After obtaining any value of the random variable $X$, for example $X = x$, we generate $Y \mid X = x \sim \text{Uniform}(x,1)$

(a) [9 points] What is the joint density of $(X,Y)$.
(b) [9 points] What is the marginal density of $Y$?

Solution:

$X \sim \text{Uniform}(0,1)$ implies

$$ f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise}, \end{cases} $$

and $Y \mid X \sim \text{Uniform}(x,1)$ implies

$$ f_{Y\mid X}(y \mid x) = \begin{cases} \frac{1}{1-x} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise}. \end{cases} $$

2 pts for $f_X(x)$. 2 pts for $f_{Y\mid X}(y \mid x)$.
(a) Then the joint density is given by

$$ f(x,y) = f_{Y\mid X}(y \mid x)f_X(x) = \begin{cases} \frac{1}{1-x} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise}. \end{cases} $$

5 pts for the joint density function.
(b) Thus,

$$ f_Y(y) = \int_0^y f(x,y)dx = \int_0^y \frac{dx}{1-x} = -\int_1^{1-y} \frac{du}{u} = -\log(1-y) $$

for $0 < y < 1$.

4 pts for knowing $f_Y(y) = \int_0^y f(x,y)dx$. 5 pts for the correct integration.
5. [18 points] [SHAI] Wires manufactured for use in a system are specified to have lengths between 12 and 14 mm. The actual wires produced by company X have a normal distribution with mean length 13 mm and standard deviation 0.5 mm.

(a) [7 points] What is the probability that a randomly selected wire from company X’s production will meet specifications?

Solution: Let $Y$ be the length of company A’s wires. Then $Y \sim \mathcal{N}(13, 0.5^2)$ and let $Z \sim \mathcal{N}(0, 1)$.

\[
P(12 < Y < 14) = P((12 - 13)/0.5 < Z < (14 - 13)/0.5) = \Phi(2) - \Phi(-2) = 0.9544 ,
\]

using the table. An answer of 95% is also acceptable.

5 pts for the conversion to standard normal. 2 pts for looking up the Z-table.

(b) [7 points] If 10 of these wires are used in each system and all are selected from company X, what is the probability that all ten in a randomly selected system will meet the specifications?

Solution: $P(\text{all 10 meet specifications}) = 0.9544^{10} = 0.6270$.

4 points for knowing that they are independent random variables and use $P(12 < y < 14)^{10}$. 3 pts for the final result.

(c) [4 points] What accuracy in manufacturing is required in order to increase the probability in (b) to at least 97%?

Solution: Need $P(\text{all 10 meet specifications}) = p^{10} \geq 0.97$. Thus $p = \Phi(c) - \Phi(-c) \geq .997$. Using the z-table, we can pick $c \approx 3$, or standard deviation less than .3 mm.

2 pts for finding $p$, 2 points for looking up the Z-table.
6. [18 points] [JUNG WUN] Suppose that \( X \) is the total time between a customer’s arrival in the store and departure from the service window, \( Y \) is the time spent in line before reaching the window, and the joint densities of these variables is given by

\[
f(x, y) = \begin{cases} 
e^{-x}, & 0 \leq y \leq x \leq \infty, \\ 0, & \text{elsewhere.} \end{cases}
\]

(a) [5 points] Find the marginal densities \( f_X(x) \) for \( X \) and \( f_Y(y) \) for \( Y \).

**Solution:** The marginals are found by “integrating out” the other variable. To wit,

\[
f_X(x) = \int_0^x f(x, y) dy = \int_0^x e^{-x} dy = xe^{-x}, \quad x \in [0, \infty),
\]

\[
f_Y(y) = \int_y^\infty f(x, y) dx = \int_y^\infty e^{-x} dx = e^{-y}, \quad y \in [0, \infty).
\]

2.5 points for each marginal density. Of which, 1.5 pts for knowing which one to integrate, 1 pt for calculation.

(b) [4 points] What is the conditional density of \( Y \) given that \( X = x \)? Be sure to specify the values of \( x \) for which this density is defined.

**Solution:**

\[
f_Y|X(y|x) = f(x, y)/f_X(x) = 1/x, \quad 0 \leq y \leq x.
\]

2 pts for knowing the definition of conditional density function. 2 points for the final result.

(c) [4 points] Are \( X \) and \( Y \) independent? Please support your answer.

**Solution:** No, they are dependent. \( X, Y \) are independent if and only if 

\( f(x, y) = f_X(x)f_Y(y) \). From (a) we can see that \( f(x, y) \neq f_X(x)f_Y(y) \).

1 point for “No”, 3 points for the reasoning.

(d) [5 points] Calculate the covariance of \( X \) and \( Y \).

**Solution:** We can integrate the marginals or compute directly:
\[ E(X) = \int_{0}^{\infty} \int_{0}^{x} xe^{-x}dydx = \int_{0}^{\infty} x^2e^{-x}dx = 2, \]
\[ E(Y) = \int_{0}^{\infty} \int_{0}^{x} ye^{-x}dydx = \int_{0}^{\infty} \frac{1}{2}x^2e^{-x}dx = 1, \]

Next, we need:
\[ E(XY) = \int_{0}^{\infty} \int_{0}^{x} xe^{-x}ydydx = \frac{1}{2} \int_{0}^{\infty} x^3e^{-x}dx = 3. \]

Therefore:
\[ COV(X,Y) = E(XY) - E(X)E(Y) = 1 \]

1 point for \( E[X] \), 1 point for \( E[Y] \), 2 points for \( E[XY] \), 1 point for covariance.